MECE 3350U
Control Systems

Lecture 11

The Root-Locus Method 1/2

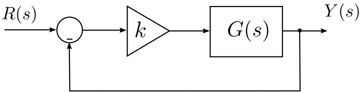
Outline of Lecture 11

By the end of today's lecture you should be able to

- Understand the influence of uncertainties in a control system
- Understand the applications of the Root-Locus method
- Apply Root-Locus method to a given system

Applications

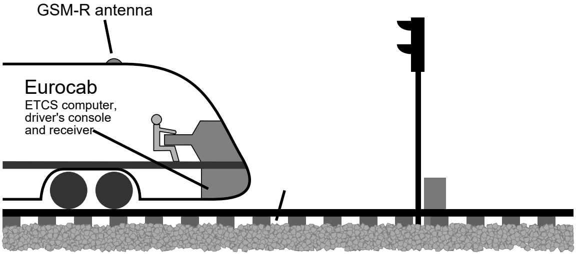
A aircraft closed-loop roll angle controller must ensure that the response time is 1 sec and the maximum overshoot is less than 15%.



What happens the performance of the controller change is the mass of the aircraft changes during the flight?

Applications

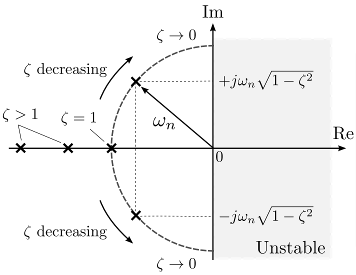
You were requested to design a cruise speed controller for a high speed train. The mechanical team requires your controller to be over-damped so that acceleration and traction is minimized.

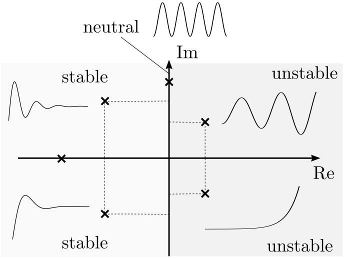


How would be controller perform if the friction between the wheels and the rail changes due to heat or snow?

Influence of pole locations

To evaluate the influence of a parameter of interest k, we would have to compute the location of the poles for different values k.

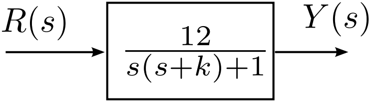




The Root-Loci method provides an alternative tool for analysis.

Influence of pole locations

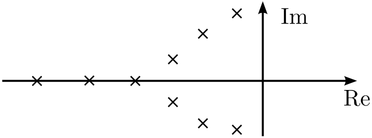
The unknown parameter *k* affects the location of poles and therefore the response of the system to an input.



What value of *k* should I choose to meet my system performance requirements?

If the value of *k* is exactly as predicted, what is the effect of a variation of *k* on my system?

Brute-force method:

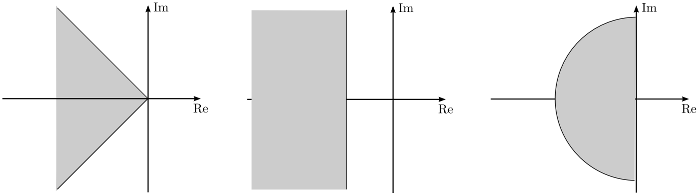


Controller design via root-locus

The damping ratio must be within a certain range.

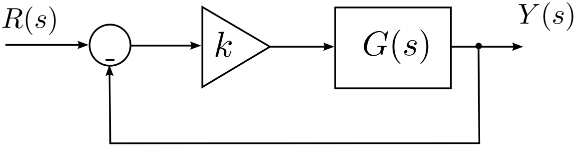
The time for exponential decay to half is specified.

The undamped frequency of oscillation is specified.



The root-locus method

Consider the following closed-loop system:



The closed-loop transfer function is

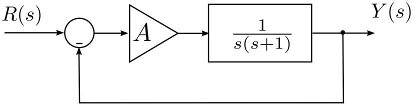
$$T(s) = \frac{kG(s)}{1 + kG(s)} \tag{1}$$

And the characteristic equations is

$$1 + kG(s) = 0 \tag{2}$$

The root-locus method

Example: How does the poles of the closed loop function change as a function of A?



$$\frac{Y(s)}{R(s)} = \frac{\frac{A}{s(s+1)}}{1 + \frac{A}{s(s+1)}}$$

$P(s) + AQ(s) = 0$

The characteristic equation is

$$1 + A\frac{1}{s(s+1)} = 0 \rightarrow 1 + A\frac{Q(s)}{P(s)} = 0$$

When $A = 0$, the poles of the closed-loop satisfy $P(s) = 0$, i.e., $p_1 = 0$, $p_2 = -1$.

When $A \rightarrow \infty$, the poles of the closed-loop satisfy $Q(s) = 0$

The root locus method

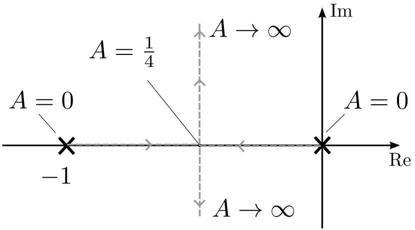
$$\frac{Y(s)}{R(s)} = \frac{\frac{A}{s(s+1)}}{1 + \frac{A}{s(s+1)}}$$

Alternatively, the characteristic equation is

$$s^2 + s + A = 0 \tag{3}$$

Using the quadratic function, the roots are

$$p_1, p_2 = -\frac{1}{2} \pm \frac{\sqrt{1-4A}}{2} \tag{4}$$



The root locus method

To analyse the influence of a given parameter of interest k, the characteristic equation must in the format

$$\boxed{1 + kH(s) = 0} \tag{5}$$

standard form for root locus analysis

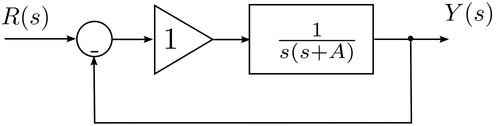
$\Rightarrow$ k is the parameter of interest

$\Rightarrow$ $H(s)$ is a function of s

The root locus is the set of values of s for which $1 + kH(s) = 0$ is satisfied as the real parameter k varies from 0 to ∞.

The root locus method

If k is the parameter of interest in the open loop transfer function, how do write the characteristic equation as $1 + kH(s) = 0$?



The characteristic equations is

$$1 + \frac{1}{s(s+A)} = 0$$

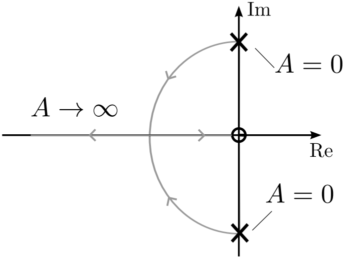
$$s^2 + As + 1 = 0; \rightarrow \overbrace{(s^2+1)}^{\text{terms without "s"}} + \underbrace{As}_{\text{terms with "s"}} = 0$$

divide the function by terms without s

$$\frac{(s^2+1)}{(s^2+1)} + A\frac{s}{(s^2+1)} = 0$$

$$1 + A\frac{s}{s^2+1} = 0$$

standard form !



Angle requirement

$$1 + kG(s) = 0, \;\; kG(s) = -1 + j0 \tag{6}$$

If the open loop transfer function is

$$G(s) = k\frac{(s+z_1)(s+z_2)(s+z_3)\dots(s+z_m)}{(s+p_1)(s+p_2)(s+p_3)\dots(s+p_n)} \tag{7}$$

The magnitude requirement for root locus is

$$|G(s)| = k\frac{|s+z_1||s+z_2||s+z_3|\dots|s+z_m|}{|s+p_1||s+p_2||s+p_3|\dots|s+p_n|} = 1 \tag{8}$$

The angle requirement for root locus is

$$\begin{aligned}\angle G(s) = \quad & \angle(s+z_1) + \angle(s+z_2) + \dots \\ & -\left[\, \angle(s+p_1) + \angle(s+p_2) + \dots \,\right] = 180^\circ + \ell 360^\circ\end{aligned}$$

where $\ell = 1, 2, 3$ ...

Angle requirement

Consider the function

$$W(s) = k\frac{s+0.4}{s^2(s+3.6)} \tag{9}$$

The root locus are the points where

$$\angle\left[k\frac{s+0.4}{s^2(s+3.6)}\right] = 180^\circ + \ell 360^\circ \tag{10}$$

which can be rewritten as

$$\angle(s+0.4) - 2\angle(s) - \angle(s+3.6) = 180^\circ + \ell 360^\circ. \tag{11}$$

Since $s = \sigma + j\omega$

$$\angle(\sigma + j\omega + 0.4) - 2\angle(\sigma + j\omega) - \angle(\sigma + j\omega + 3.6) = 180^\circ + \ell 360^\circ \tag{12}$$

and $\angle s = \tan^{-1}(\omega/\sigma)$, the root locus function is

$$\tan^{-1}\left(\frac{\sigma}{\omega+0.4}\right) - 2\tan^{-1}\left(\frac{\sigma}{\omega}\right) - \tan^{-1}\left(\frac{\sigma}{\omega+3.6}\right) = 180^\circ + \ell 360^\circ \tag{13}$$

10 rules for drawing the root-locus

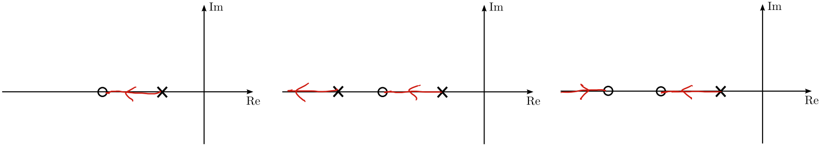
Rule 1: As k varies from 0 to ∞, there are n lines (loci) where n is the degree of $Q(s)$ or $P(s)$, whichever is greater.

$$1 + k\overset{H(s)}{\boxed{\frac{Q(s)}{P(s)}}} = 0$$

Poles move toward zeros!

Rule 2: As k varies from 0 to ∞, the **roots of the characteristic equation** move from the poles of $H(s)$ (when $P(s) = 0$) to the zeros of $H(s)$ ($Q(s) = 0$).

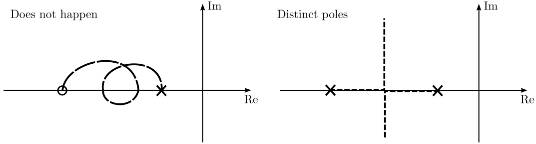
$$\boxed{P(s) + kQ(s) = 0, \qquad \frac{1}{k}P(s) + Q(s) = 0}$$



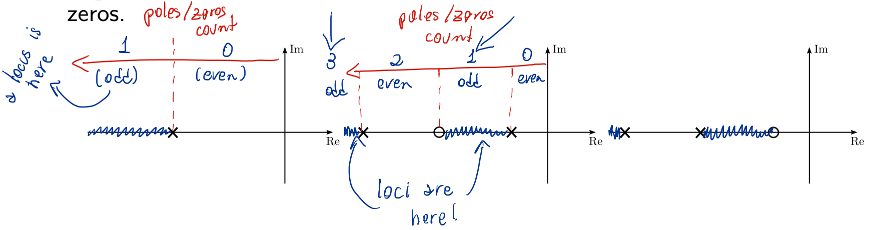
10 rules for drawing the root-locus

Rule 3: The root loci must be symmetrical with respect to the horizontal axis.

Rule 4: The a root cannot cross its path

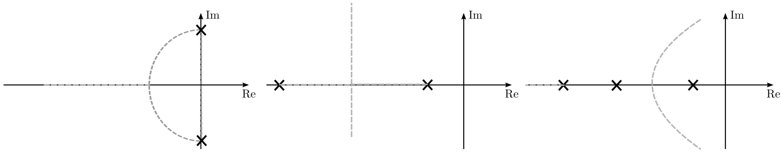


Rule 5: The loci are on the real axis to the left of an odd number of poles and zeros.



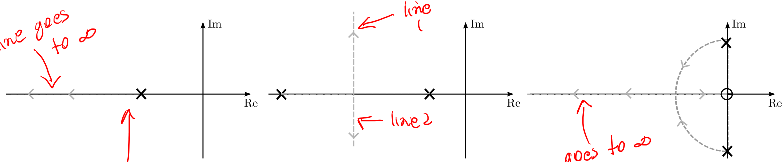
10 rules for drawing the root-locus

Rule 6: Lines leave (break out) and enter (break in) the real axis at 90°



Rule 7: If there a different number of poles (n) and zeros (m), extra lines that do not have a pair go to or come from infinity.

→ $n-m$ lines go to ∞.



line goes to ∞

line

line 2

goes to ∞

$n-m=1$

$n-m=2$

2 lines → ∞

$n-m=1$

10 rules for drawing the root-locus

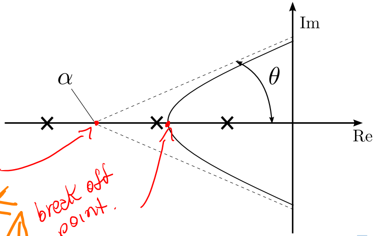
Rule 8: The angle of the asymptotes of the curves that go to infinity is

$$\theta = \frac{180^\circ + 360^\circ(q-1)}{n-m}, \; q = 1, 2, \ldots, n-m \tag{14}$$

n is the order $P(s)$, m is the order of $Q(s)$ thus $n-m$ is the number of unmatched poles.

The asymptote radiates out from the point $s = \alpha$ on the real axis where

$$\alpha = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} \tag{15}$$



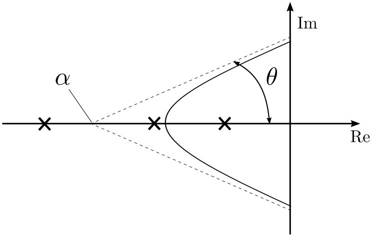
centre of asymptotes ≠ break off point.

2

α is the centroid of the asymptotes and NOT the break off point from the real axis.

10 rules for drawing the root-locus

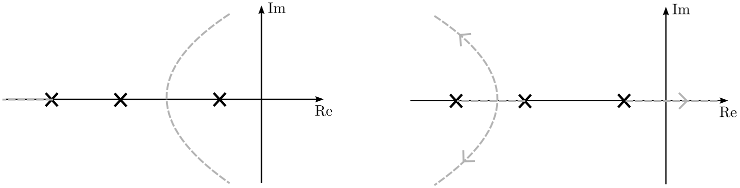
Rule 9: If there are a least two lines that go to infinity, then the sum of all the roots is constant.



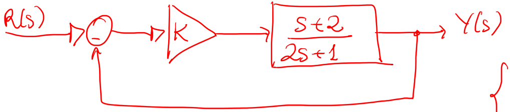
Practical applications: As k increases, the real root moves to the left twice as fast as the conjugate roots approach the imaginary axis.

10 rules for drawing the root-locus

Rule 10: If the k sweeps from 0 to $-\infty$, the root loci can be drawn by reversing Rule 5 **and** adding a 180° to the asymptote angles.



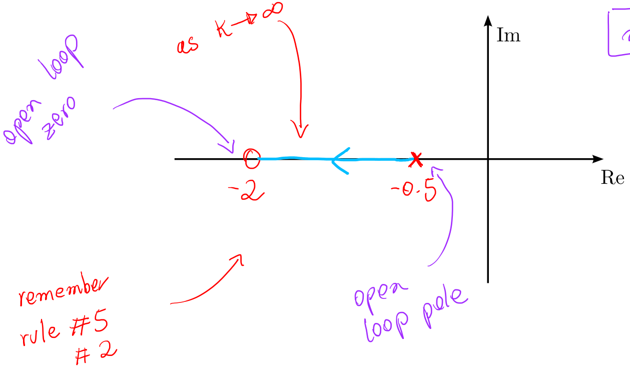
Exercise 50



{ O zeros
X poles

Determine the root-locus plot for the following transfer function.

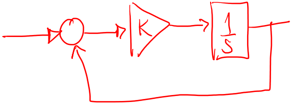
$$G(s) = k\frac{s+2}{2s+1} \tag{16}$$



$\boxed{n-m=0}$

No locus goes to ∞.

Exercise 51



Determine the root-locus plot for the following transfer function.

$n - m = 1$

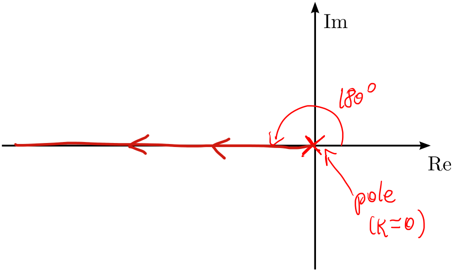
$\hookrightarrow$ 1 line goes to ∞

$q = 1$

$\theta = \dfrac{180^0 + 360(q-1)}{1}$

$\boxed{\theta = 180^0}$

$$G(s) = \frac{K}{s} \tag{17}$$



Exercise 52

Determine the root-locus plot for the following transfer function.

$$G(s) = \frac{K}{(2s+1)(s+1)} \quad (18)$$

$n - m = 2$

$q = 1, 2.$

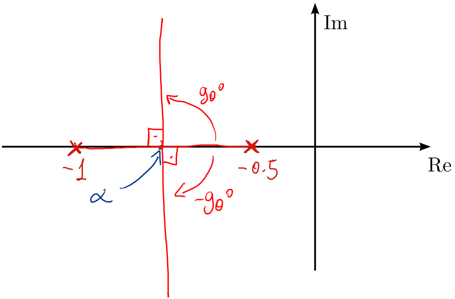
$q = 1$

$$\theta_1 = \frac{180 \pm 360(q-1)}{n-m}$$

$\theta_1 = 90^\circ$

$q = 2$

$\theta_2 = 270^\circ$ (or -90°)



$$\alpha = \frac{\sum p - \sum z}{n-m}$$

$$\alpha = \frac{-0.5 - 1 - 0}{2}$$

$$\boxed{\alpha = -0.75}$$

centre of asymptotes

Exercise 53

$n = 4$
$m = 2$

O zeros
X poles

Determine the root-locus plot for the following transfer function.

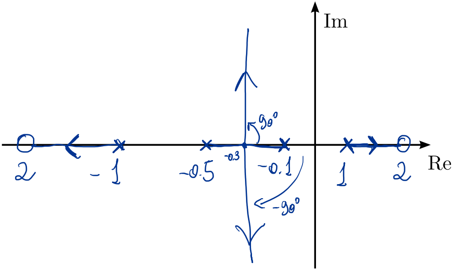
$$G(s) = k\frac{(s+2)(s-2)}{(s+1)(s+0.5)(s+0.1)(s-1)} \tag{19}$$

$n - m = 2$

$$\theta = \frac{180^\circ + 360^\circ(q-1)}{2}$$

$q = 1$
$\theta = 90^\circ$

$q = 2$
$\theta = -90^\circ$



$$\alpha = \frac{\sum p - \sum z}{n - m}$$

$$\alpha = \frac{(-1 - 0.5 - 0.1 + 1) - (2 - 2)}{2} = -0.3$$

Exercise 54

$n - m = 3$

$\rightarrow$ 3 lines go to ∞ $\rightarrow$ $q = 1, 2, 3$

Determine the root-locus plot for the following transfer function.

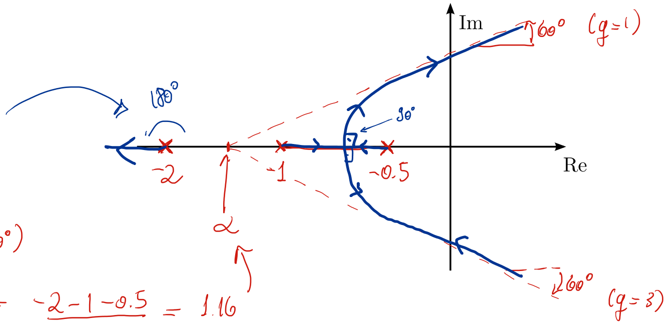
$\theta = \dfrac{180^\circ + 360^\circ (q-1)}{n-m}$

$$G(s) = \frac{K}{(2s+1)(s+1)(0.5s+1)} \tag{20}$$

$q = 1$
$\theta_1 = 60^\circ$

$q = 2$
$\theta_2 = 180^\circ$

$q = 3$
$\theta_3 = 300^\circ$
$(\text{or } -60^\circ)$



$\alpha = \dfrac{-2-1-0.5}{3} = 1.16$

Exercise 55

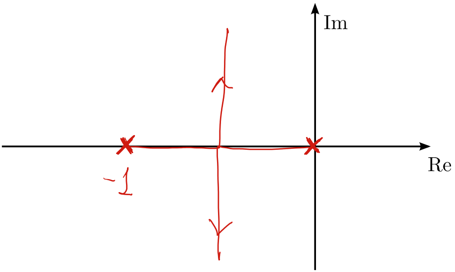
Determine the root-locus plot for the following transfer function.

$\theta_1 = 90^\circ$

$\theta_2 = -90^\circ$

$\alpha = 0.5$

$$G(s) = \frac{K}{s(s+1)} \tag{21}$$



Exercise 56

Homework

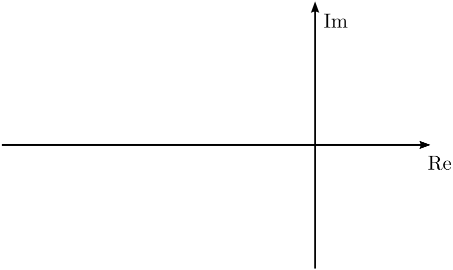
check the solution with Matlab

Determine the root-locus plot for the following transfer function.

$$G(s) = \frac{K}{s(s+1)(s+2)} \tag{22}$$

Matlab code:

```
s= tf([1 0],[1])
H= 1/(s*(s+1)*(s+2))
rlocus (H)
```

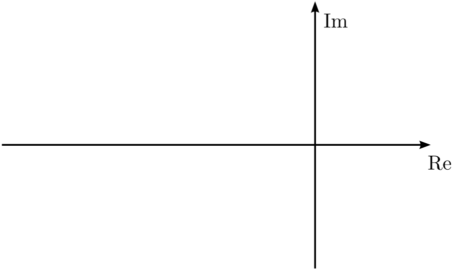


Exercise 57

Homework

Determine the root-locus plot for the following transfer function.

$$G(s) = k\frac{(s+3)(s+4)}{(s+1)(s+2)} \tag{23}$$



Solution here

https://goo.gl/Fq14E4

Exercise 58

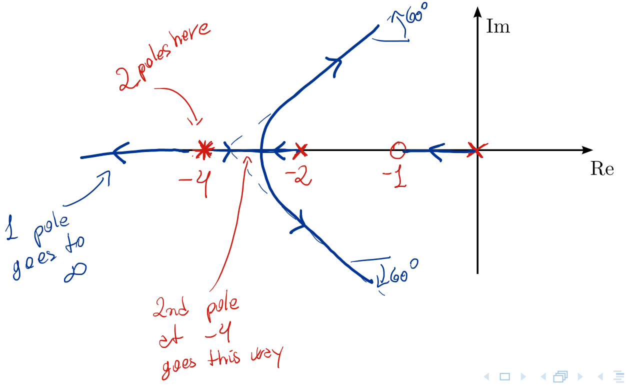
$n - m = 4 - 1 = 3$

Determine the root-locus plot for the following transfer function.

Make it 2

$$G(s) = K\frac{(s+1)}{s(s+2)\underbrace{(s+4)^2}_{}} \tag{24}$$

double pole



$\theta_1 = 60°$

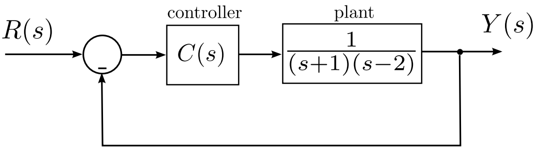
$\theta_2 = 180°$

$\theta_3 = 300°$ (or $-60°$)

Exercise 59

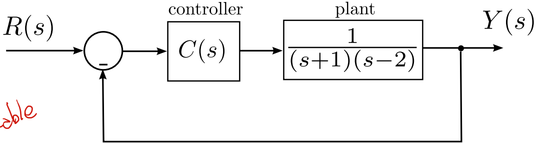
Homework

A controller for closed-loop feedback system is to be designed to stabilize the system shown. In order to simply the controller, the electrical team recommended that a proportional controller should be used.

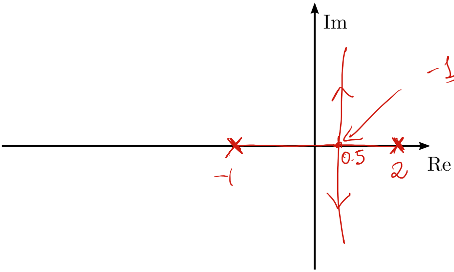


Can the system be stabilized if $G(s) = k$, with $0 \leq k < \infty$?

Exercise 59 - continued



Show that the system is unstable $\forall K > 0$



$\frac{-1+2}{2} = 0.5$

there will always be a pole in the right half plane.
→ Unstable $\forall K$:(

Next class...

- More on the root locus method