MECE 3350U
Control Systems

Lecture 17
Bode Plots 2/2

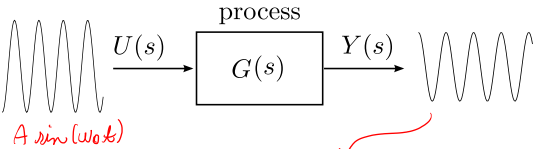
Outline of Lecture 17

By the end of today's lecture you should be able to

- Represent magnitude and phase in a Bode plot
- Draw the Bode plot for functions having complex poles

Frequency response

Frequency response: The steady-state response of the system to a sinusoidal input signal.



$$y(t) = AM\sin(\omega_0 t + \phi) \tag{1}$$

where A the amplitude of the input signal and

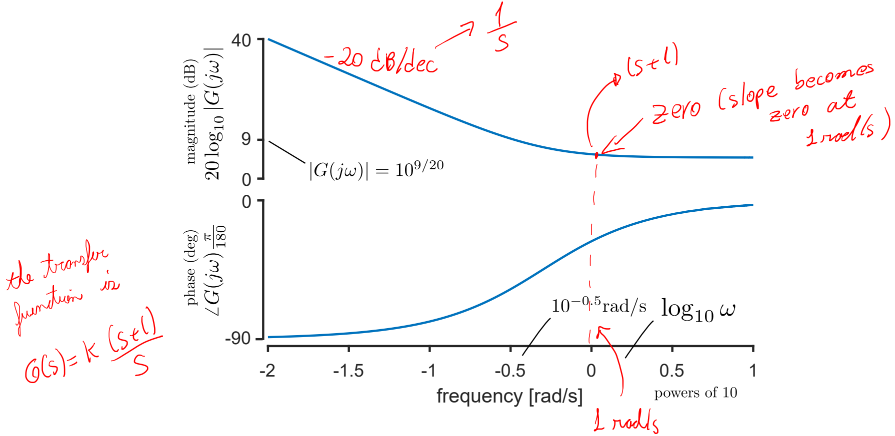
$$M = |G(j\omega_0)| = |G(s)|_{s=j\omega_0} = \sqrt{[\Re G(j\omega_0)]^2 + [\Im G(j\omega_0)]^2}$$

$$\angle G(j\omega_0) = \phi = \tan^{-1}\left(\frac{\Re[G(j\omega_0)]}{\Im[G(j\omega_0)]}\right)$$

Bode plots

The vertical axis shows the phase ϕ and gain $20\log(G(j\omega))$

The horizontal axis is logarithmic $\log_{10}(\omega)$



Bode plots - review

Given a transfer function

$$G(s) = k\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)} \tag{2}$$

The gain is

$$[G(j\omega)| = \; 20\log\left[k\frac{\prod_{i=1}^{n}(j\omega+z_i)}{\prod_{i=1}^{m}(j\omega+p_i)}\right] \tag{3}$$

Since $\log(a\times b) = \log(a) + \log(b)$, we can rewrite the gain as

$$[G(j\omega)| = 20\log(k) + \sum_{i=1}^{n}\left[20\log(j\omega+z_i)\right] + \sum_{i=1}^{m}\left[20\log\frac{1}{(j\omega+p_i)}\right] \tag{4}$$

Thus, if we know the Bode plot of basic functions, we can sketch the Bode diagram of $G(s)$.

Bode plots - review

Given a transfer function

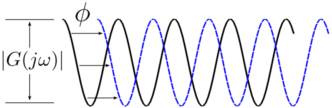
$$G(s) = k\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)} \tag{5}$$

The phase is

$$\angle[G(j\omega)| = \phi = \tan^{-1}\left[\frac{\Im[G(j\omega)]}{\Re[G(j\omega)]}\right] \tag{6}$$

$$\phi = \angle(k) + \sum_{i=1}^{n}[\angle(j\omega + z_i)] + \sum_{i=1}^{m}\left[\angle\frac{1}{j\omega + p_i}\right] \tag{7}$$

Thus, if we know the Bode plot of basic functions, we can sketch the Bode diagram of $G(s)$.



Bode plot building blocks - review

1 - Constant gain

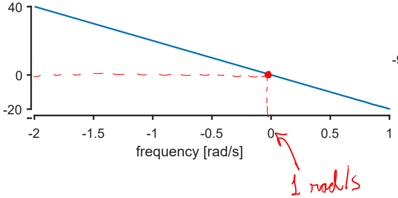
→ Gain: $|k|$ or $20\log(|k|)$

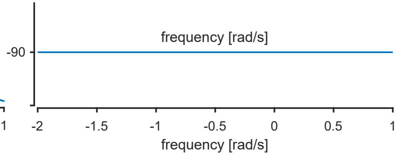
→ Phase: $\phi = 0\ \forall\ \omega$ if $k > 0$, -180° otherwise

2 - Pole at the origin

→ Gain: $-20\log(\omega)$

→ Phase: $-90^\circ\ \forall\ \omega$



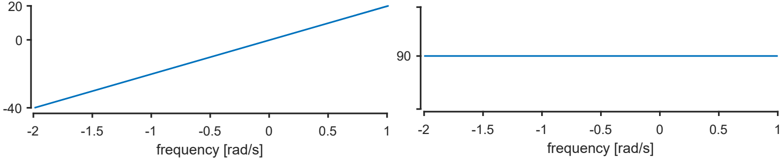


Bode plot building blocks - review

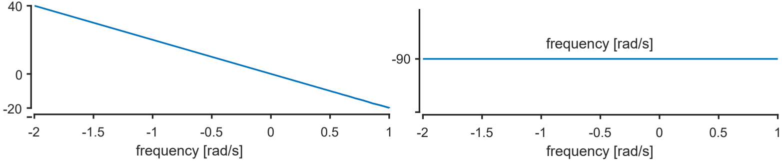
3 - Zero at the origin

→ Gain: $20\log(\omega)$

→ Phase: $90^\circ\ \forall\ \omega$



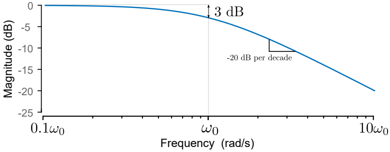
Note that this is the negative of a pole at the origin:

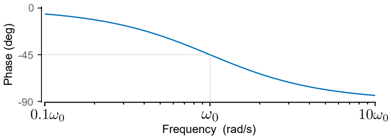


Bode plot building blocks - review

4 - Real pole: $G(s) = \frac{1}{\frac{s}{\omega_0}+1}$, $\omega_0 \in \Re^*$

	$\omega << \omega_0$	$\omega = \omega_0$	$\omega = \omega_0$
Gain	0	-3 dB	$-20 \log \sqrt{\frac{\omega}{\omega_0}}$
Phase	0	-45°	-90°

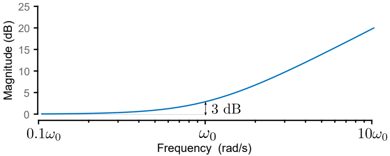


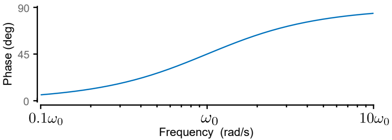


Bode plot building blocks - review

5 - Real zero: $G(s) = \frac{s}{\omega_0} + 1$, $\omega_0 \in \Re^*$

	$\omega << \omega_0$	$\omega = \omega_0$	$\omega = \omega_0$
Gain	0	+3 dB	$20 \log \sqrt{\frac{\omega}{\omega_0}}$
Phase	0	45°	90°





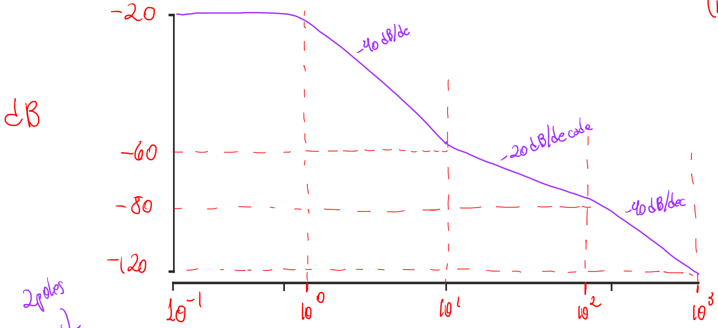
The real zero is the negative of a real pole on the Bode plot

Exercise 100

Draw the approximate Bode plot of the transfer function

$$G(s) = \frac{(s+10)}{(s+1)^2(s+100)} = \frac{10\left(\frac{s}{10}+1\right)}{(s+1)^2\,100\left(\frac{s}{100}+1\right)} = \frac{0.1\left(\frac{s}{10}+1\right)}{(s+1)^2\left(\frac{s}{100}+1\right)}$$

$20\log(0.1) = -20$



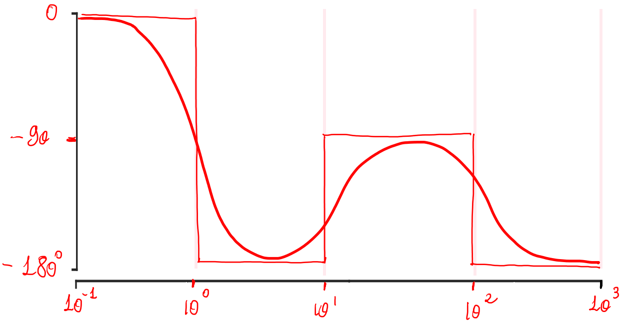
2 poles

cut off frequencies 1, 10, 100 rad/s

Exercise 100 - continued

Draw the approximate Bode plot of the transfer function $G(s) = \dfrac{0.1\left(\frac{s}{10}+1\right)}{(s+1)^2\left(\frac{s}{100}+1\right)}$

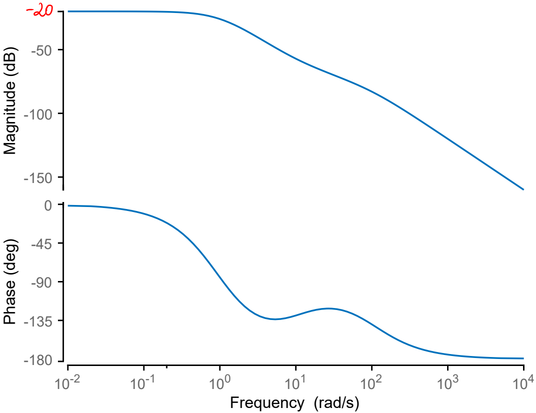
$$G(s) = \frac{(s+10)}{(s+1)^2(s+100)}$$



Exercise 100 - continued

bode(H)

Result using Matlab



Bode plot building blocks

6 - Complex conjugate poles $G(s) = \frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$ $= \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + \frac{2\zeta s}{\omega_0} + 1}$

The equation can be rearranged as

$$G(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1} \quad \rightarrow \quad G(j\omega) = \frac{1}{-\left(\frac{\omega}{\omega_0}\right)^2 + 1 + j2\zeta\left(\frac{\omega}{\omega_0}\right)} \tag{8}$$

$$G(j\omega) = \frac{1}{-\left(\frac{\omega}{\omega_0}\right)^2 + 1 + j2\zeta\left(\frac{\omega}{\omega_0}\right)} \times \frac{-\left(\frac{\omega}{\omega_0}\right)^2 + 1 - j2\zeta\left(\frac{\omega}{\omega_0}\right)}{-\left(\frac{\omega}{\omega_0}\right)^2 + 1 - j2\zeta\left(\frac{\omega}{\omega_0}\right)} \tag{9}$$

$$G(j\omega) = \frac{1-\left(\frac{\omega}{\omega_0}\right)^2}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2} + j\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2} \tag{10}$$

Bode plot building blocks

$$G(j\omega) = \frac{1-\left(\frac{\omega}{\omega_0}\right)^2}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2} + j\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}$$

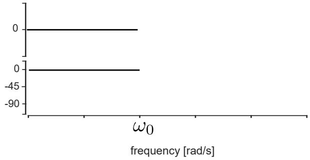
Case 1: $\omega << \omega_0$,

→ Thus: $\omega/\omega_0 \approx 0$ and $G(j\omega)$ simplifies to

$$G(j\omega) \approx 1 + 0j$$

→ The gain is $20\log(\sqrt{1^2+0^0}) = 0$ dB

→ The phase is $\phi = \tan^{-1}(\frac{0}{1}) = 0°$



Bode plot building blocks

$$G(j\omega) = \frac{1-\left(\frac{\omega}{\omega_0}\right)^2}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2} + j\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}$$

Case 2: $\omega >> \omega_0$

$\rightarrow$ Thus $G(j\omega)$ simplifies to

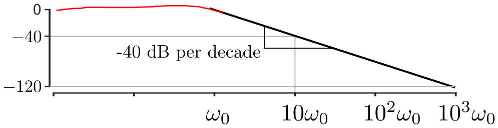
$$G(j\omega) \approx -\frac{1}{\left(\frac{\omega}{\omega_n}\right)^2} - j\frac{2\zeta}{\left(\frac{\omega}{\omega_0}\right)^3}$$

$\rightarrow$ The gain is

$$G = 20\log\sqrt{\left[-\left(\frac{\omega}{\omega_n}\right)^{-2}\right]^2+\left[-2\zeta\left(\frac{\omega}{\omega_0}\right)^{-3}\right]^2} \approx 20\log\left(\frac{\omega}{\omega_0}\right)^{-2}$$

$$\boxed{G = -40\log\left(\frac{\omega}{\omega_0}\right)}$$

Bode plot building blocks

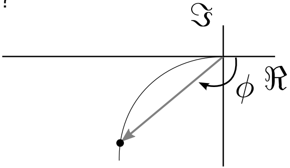


Still when $\omega >> \omega_0$, let us look at the phase:

$$G(j\omega) \approx -\frac{1}{\left(\frac{\omega}{\omega_0}\right)^2} - j\frac{2\zeta}{\left(\frac{\omega}{\omega_0}\right)^3}$$

$$\phi = \tan^{-1}\left[\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)^{-3}}{-\left(\frac{\omega}{\omega_0}\right)^{-2}}\right] = \tan^{-1}\underbrace{\left[2\zeta\frac{\omega_n}{\omega}\right]}_{\to 0} = -180^\circ$$

Why -180° instead of 0°?



Bode plot building blocks

$$G(j\omega) = \frac{1 - \left(\frac{\omega}{\omega_0}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2} + j\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)}{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}$$

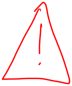
Case 3: $\omega = \omega_0$

$\rightarrow$ Thus $G(j\omega)$ simplifies to

$$G(j\omega) = 0 - j\left(\frac{1}{2\zeta}\right)$$

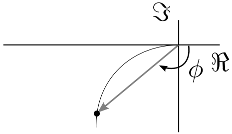
$\rightarrow$ The gain is

$$G = 20\log\sqrt{\left(\frac{-1}{2\zeta}\right)^2} = 20\log(2\zeta)^{-1} = -20\log(2\zeta)$$



$\zeta = 0.5,\ G = 0$ dB
$\zeta < 0.5$, there is a peak at $\omega = \omega_0$
$\zeta > 0.5$, there is a negative gain at $\omega = \omega_0$
$\zeta = 0,\ \mathrm{G}(\omega = \omega_0) \rightarrow \infty$

Bode plot building blocks



When $\omega = \omega_0$, the phase is

$$G(j\omega) = 0 - j\left(\frac{1}{2\zeta}\right)$$

$$\phi = \tan^{-1}\left(-\frac{\frac{1}{2\zeta}}{c}\right)_{c\to 0} = -90^\circ$$

In summary

	$\omega << \omega_n$	$\omega >> \omega_n$	$\omega = \omega_n$
Gain	0	-40 dB/decade	$-20\log(2\zeta)$
Phase	0°	-180°	-90°

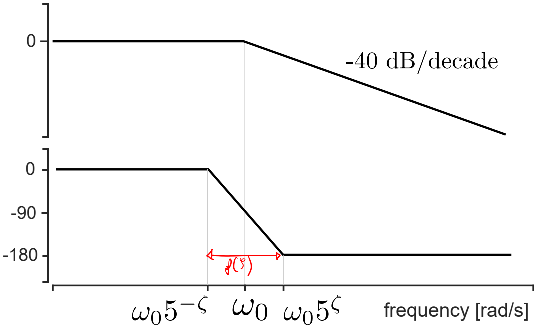
Notice that this analogous to having two equal real poles.

with the # of

Bode plot building blocks

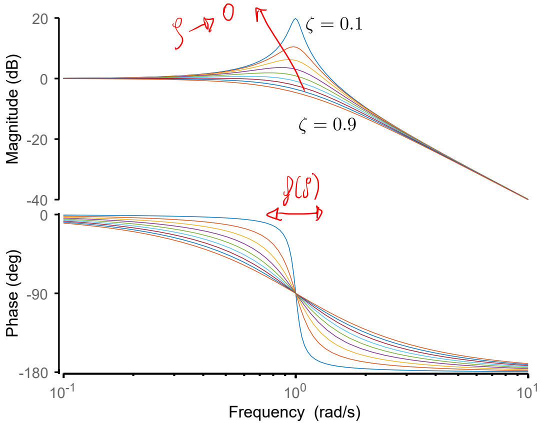
Summary - Bode plots for complex poles

	$\omega << \omega_n$	$\omega >> \omega_n$	$\omega = \omega_n$
Gain	0	-40 dB/decade	$-20\log(2\zeta)$
Phase	$0°$	-180°	$-90°$



Influence of the damping ratio

$$G(s) = \frac{1}{s^2 + 2\zeta s + 1}$$



Resonance frequency

The frequency at which the gain reaches its maximum value is called the **resonance frequency**.

The resonance frequency satisfies $\omega = \omega_r$

$$\frac{\partial}{\partial \omega}\sqrt{\left(\frac{1-\left(\frac{\omega}{\omega_0}\right)^2}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}\right)^2+\left(\frac{-2\zeta\left(\frac{\omega}{\omega_0}\right)}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_0}\right)\right]^2}\right)^2}=0$$

$$\omega_r = \omega_0\sqrt{1-2\zeta^2}, \text{ for } \zeta < \frac{\sqrt{2}}{2} \tag{11}$$

Thus, the maximum value M_ω of $|G(j\omega)|$ is

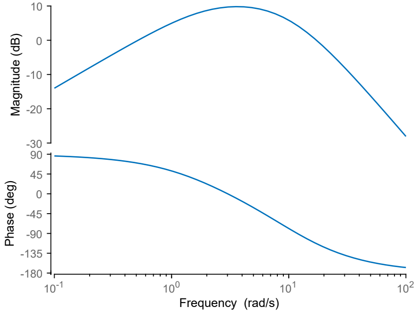
$$M_\omega = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, \text{ for } \zeta < \frac{\sqrt{2}}{2} \tag{12}$$

Exercise 101

The frequency response of a dynamic system has many practical applications and is often used in order to estimate the system parameters. Knowing that a system transfer function is

$$G(s) = k\frac{s}{(s+a)(s^2+20s+100)}$$

And its frequency response is shown in the Bode plot, determine k and a.



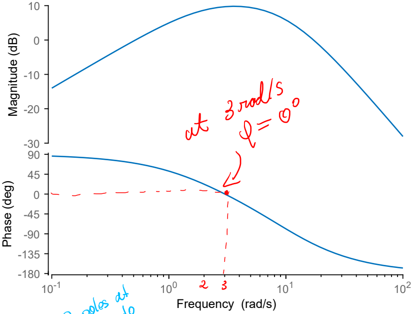
Exercise 101 - continued

$$G(s) = K\frac{s}{(s+a)(s^2+20s+100)}$$

$$G(s) = \frac{Ks}{(s+a)(s+10)^2}$$

$$G(j\omega) = \frac{Kj\omega}{(j\omega+a)(j\omega+10)^2}$$

$$\phi = \tan^{-1}\left(\frac{Im}{Re}\right)$$



zero at the origin ↓ pole "a" ↓ 2 poles at -10

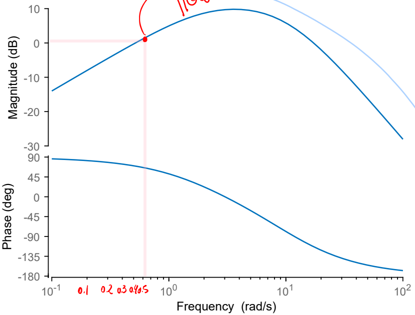
$$\phi = +90^\circ - \text{atan}\left(\frac{\omega}{a}\right) - 2\,\text{atan}\left(\frac{\omega}{10}\right)$$

$$-90^\circ + 2\,\text{atan}\left(\frac{3}{10}\right) = -\text{atan}\left(\frac{3}{a}\right)$$

solving for "a" → $\boxed{a = 2}$

Exercise 101 - continued

$$G(s) = \frac{K\,s}{(s+2)(s+10)^2}$$



$$|G(j\omega)| = 20\log\left(K\frac{s}{(s+2)(s+10^2)}\right) = 20\log\left(K\frac{j\omega}{(j\omega+2)(j\omega+10)^2}\right)$$

$$|G(j\omega)| = 20\log(K) + 20\log(\omega) - 20\log\sqrt{2^2+\omega^2} - 2\times 20\log\sqrt{10^2+\omega^2}$$

$$0 = 20\log(K) + 20\log(0.5) - 10\log(2^2+0.5^2) - 20\log(10^2+0.5^2)$$

$$\boxed{K \simeq 400}$$

Exercise 102

A low-pass filter is a filter that passes signals with a frequency lower than a certain cut-off frequency and attenuates signals with frequencies higher than the cut-off frequency. A hypothetical filter has the transfer function

$$G(s) = \frac{4}{(s^2 + 0.4s + 4)}$$

Sketch its frequency response.

Exercise 102 - continued

$$G(s) = \frac{4}{(s^2 + 0.4s + 4)}$$

$$G(s) = \frac{1}{\left(\frac{s}{2}\right)^2 + 0.2\left(\frac{s}{2}\right) + 1}$$

$$\omega_0 = 2$$

$$2\zeta = 0.2$$

$$\zeta = 0.1$$

when $\omega = \omega_0$

$$G(j\omega) = -20 \log(2\zeta)$$

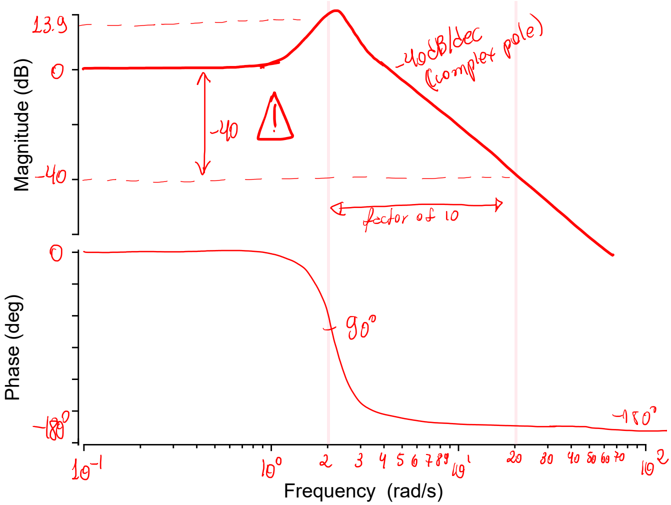
$$G(0.1) = -20 \log(0.1)$$

$$G(0.1) = +13.9 \text{ dB}$$

Exercise 102 - continued

$\omega_0 = 1$ rad/s and $\omega_0 = 2$ rad/s (complex poles).

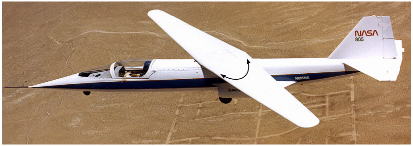
$$G(s) = \frac{1}{\left(\frac{s}{2}\right)^2 + 0.2\left(\frac{s}{2}\right) + 1}$$



Exercise 103

The experimental oblique wing aircraft has a wing that pivots. Its control system loop transfer function is

$$G(s) = \frac{4(0.5s + 1)}{s(2s+1)\left[\left(\frac{s}{8}\right)^2 + \frac{s}{20} + 1\right]}$$



Sketch its frequency response.

Exercise 103 - continued

$$G(s) = \frac{4(0.5s + 1)}{s(2s+1)\left[\left(\frac{s}{8}\right)^2 + \frac{s}{20} + 1\right]}$$

$$G(s) = \frac{4(0.5)(s+2)}{s(2)(s+0.5)\left[\left(\frac{s}{8}\right)^2 + \frac{s}{20} + 1\right]} \rightarrow G(s) = \frac{4(0.5)2\left(\frac{s}{2}+1\right)}{s\,2(0.5)\left(\frac{s}{0.5}+1\right)\left[\left(\frac{s}{8}\right)^2 + \frac{s}{20} + 1\right]}$$

$$G(s) = \frac{4\left(\frac{s}{2}+1\right)}{s\left(\frac{s}{0.5}+1\right)\left[\left(\frac{s}{8}\right)^2 + \frac{s}{20} + 1\right]}$$

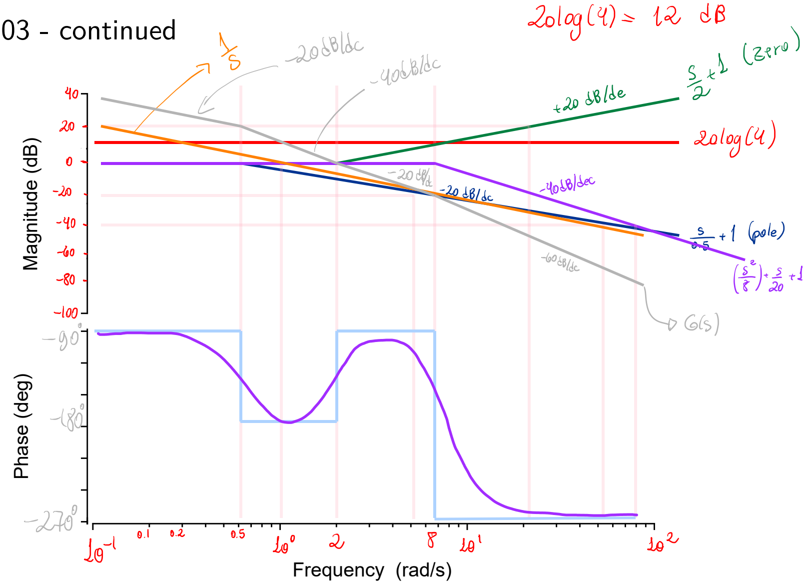
cut off frequencies: $\omega_1 = 0.5$ rad/s (real pole)

$\omega_2 = 2$ rad/s (real zero)

$\omega_3 = 8$ rad/s (complex pole)

→ pole at the origin

Exercise 103 - continued



Exercise 104 - Matlab problem

Consider the closed-loop transfer function

Homework

$$R(s) = \frac{30}{s^2 + s + 30}$$

Develop a Matlab code to obtain the Bode plot and verity that the resonant frequency is 5.44 rad/s and that the peak magnitude is 14.8 dB.

Compare the results of you code with the results of "Bode(R)" function.

Next class...

- Stability in the frequency domain