MECE 3350U
Control Systems

Lecture 20

Stability Margins

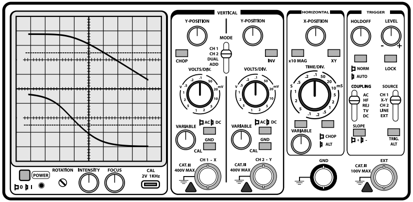
Outline of Lecture 20

By the end of today's lecture you should be able to

- Calculate the gain and phase margin of a system
- Obtain the gain and phase margin from a Bode plot
- Quantify the stability of an open-loop transfer function

Applications

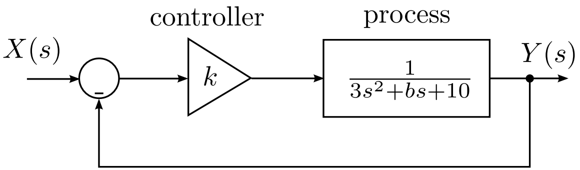
We wish to develop a closed-loop controller for a system whose dynamics is unknown. The frequency response of the open-loop system has been obtained experimentally using an oscilloscope.



What does it tell us about its closed-loop stability?

Applications

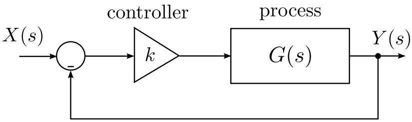
The controller gain k has been specified to the process shown.



If b changes during operation, how can we ensure that the system remains stable?

Bode *vs* Nyquist plots

The closed loop system



if $1 + KG(s) = 1$

$|T(j\omega)| \rightarrow \infty$

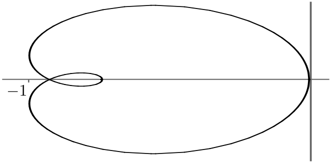
$$T(s) = \frac{kG(s)}{1 + kG(s)}$$

if $\boxed{KG(s) = -1}$

point that must be avoided on the Nyquist plot

might be stable for only a range of values of k.

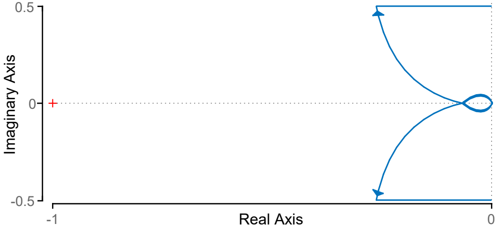
The proximity of the $L(j\omega)$ locus to $-1 + j0$ is a measure of the relative stability of the system.



Bode *vs* Nyquist plot

Consider the open-loop transfer function

$$L(j\omega) = \frac{k}{j\omega(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)}$$

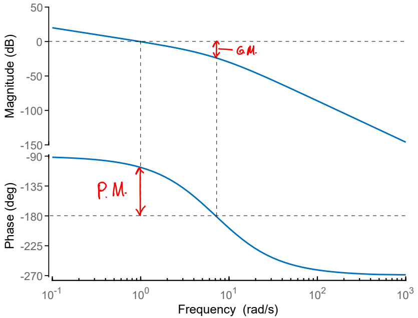


As k is increased, the Nyquist plot approaches $-1 + 0j$ and eventually encircles the 1 point.

The point $-1 + 0j$ can also be expressed in polar form as $1\angle -180^\circ$

Bode *vs* Nyquist plot

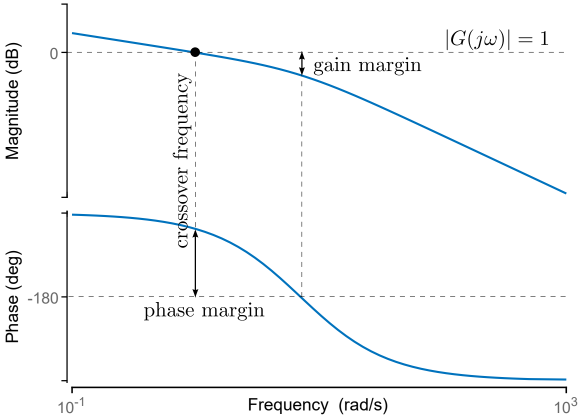
$$L(j\omega) = \frac{k}{j\omega(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)}$$



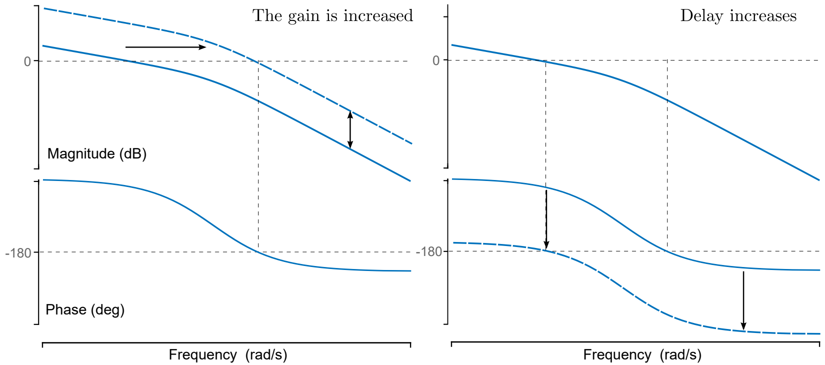
Gain margin: The increase in the loop gain when $\phi = -180^\circ$ that results in $|L(j\omega)| = 1$ or 0 dB.

Phase margin: The amount of phase shift at the crossover frequency that results in $\angle L(j\omega) = -180^\circ$.

Gain and phase margins



Gain and phase margins

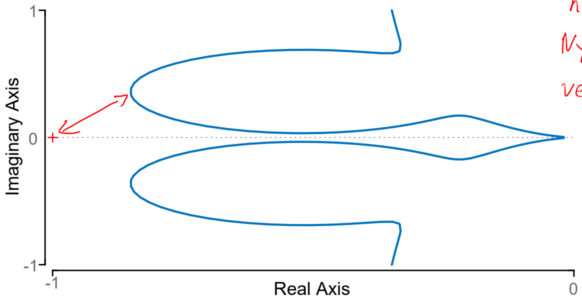


True or false?

The following open-loop transfer function is closed-loop stable for any $k > 0$.

$$L(s) = k\frac{s^2 + 0.1s + 0.5}{s(s+1)(s^2 + 0.05 + 0.5)}$$

true!

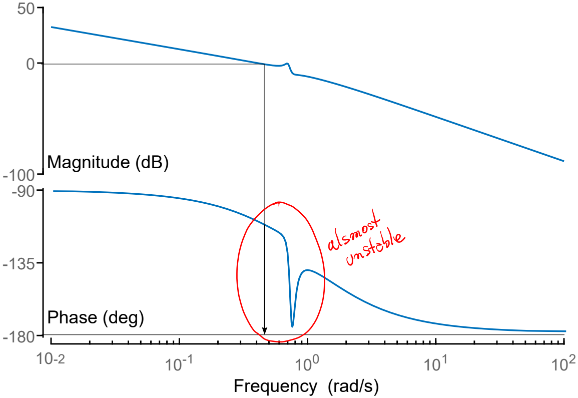


however the Nyquist plot is very close to -1 and can be unstable in practice.

True or false?

The following open-loop transfer function is closed-loop stable for any $k > 0$.

$$L(s) = k\frac{s^2 + 0.1s + 0.5}{s(s+1)(s^2 + 0.05 + 0.5)}$$



Phase margin

As an example, consider the open-loop second-order system

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \rightarrow \frac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)} \quad (1)$$

Step 1 - Find the crossover frequency (0 dB)

At the crossover frequency $\omega = \omega_c$, the magnitude is 1. Find ω_c that gives

$$\frac{\omega_n^2}{\omega_c\sqrt{\omega_c^2 + 4\zeta^2\omega_n^2}} = 1.$$

Step 2 - Find the phase of $G(j\omega)$ at ω_c for ω_c found in Step 1, i.e. $\angle G(j\omega_c)$

$$\phi = -90^\circ - \tan\left(\frac{\omega_c}{2\zeta\omega_n}\right)$$

Step 3 - The margin phase is $180 - |\phi|$

Gain margin

Consider the same open-loop second-order system

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} \rightarrow \frac{\omega_n^2}{j\omega(j\omega+2\zeta\omega_n)}$$

Step 1 - Find the frequency ω_f where $\angle|G(j\omega)| = -180^\circ$

$$G(j\omega) = \frac{\omega_n^2}{j\omega(j\omega+2\zeta\omega_n)} = \frac{\omega_n^2}{-\omega_f^2 + j2\zeta\omega_n\omega_f} \times \frac{-\omega_f^2 - j2\zeta\omega_n\omega_f}{-\omega_f^2 - j2\zeta\omega_n\omega_f}$$

$$G(j\omega) = -\frac{\omega_n^2\omega_f^2}{\omega_f^4 + 4\zeta^2\omega_n^2\omega_f^2} - j\frac{2\zeta\omega_n^3\omega_f}{\omega_f^4 + 4\zeta^2\omega_n^2\omega_f^2}$$

At ω_f, $\Im[G(j\omega_f)] = 0$ (imaginary part is zero)

$$-\frac{2\zeta\omega_n^3\omega_f}{\omega_f^4 + 4\zeta^2\omega_n^2\omega_f^2} = 0$$

$\omega_f = 0$ Not a valid frequency

$\omega_f = \infty$ What does it mean? $\phi \neq -180^\circ \;\; \forall \; K$

$\omega_f =$ constant. Proceed to Step 2

Gain margin

Step 2 - Find the gain of $G(j\omega)$ at $\omega = \omega_c$, i.e., $|G(j\omega_f)|$

$$k_{MG} = \frac{\omega_n^2}{\omega_f \sqrt{\omega_f^2 + 4\zeta^2\omega_n^2}}$$

Then gain margin in Decibels is

$$MG = -20\log(G)$$

$\rightarrow$ $MG > 0$: The current gain can be multiplied by k_{MG} **dB** before the system becomes marginally stable (or MG decibels can be added before instability);

$\rightarrow$ $MG < 0$ The gain can be divided by k_{MG} dB before the system becomes marginally stable (or MG decibels can be subtracted before instability).

$\rightarrow$ $MG = 0$ The system is marginally stable.

Exercise 117

A unit feedback control system has a loop transfer function

$$L(s) = \frac{k}{s(s+2)(s+10)}$$

For $k = 50$, determine the cross over frequency, the gain margin, and the phase margin.

Exercise 117 - continued

$$L(s) = \frac{k}{s(s+2)(s+10)} \rightarrow \frac{50}{j\omega(j\omega+2)(j\omega+10)}$$

at the crossover frequency.

$|L(j\omega)| = 1$ (or 0dB) $\rightarrow \dfrac{50}{\omega\sqrt{\omega^2+2^2}\sqrt{\omega^2+10^2}} = 1$

crossover frequency

$50 = \omega\sqrt{\omega^2+2^2}\sqrt{\omega^2+10^2} \rightarrow 2500 = \omega^2(\omega^2+4)(\omega^2+100)$

$\boxed{\omega = \omega_c = 1.82 \text{ rad/s}}$

Phase margin (at ω_c)

$\phi = 0 - 90^\circ - \tan^{-1}\left(\frac{1.82}{2}\right) - \tan^{-1}\left(\frac{1.82}{10}\right)$

$\boxed{\phi = -142^\circ}$

$P.M = -180 - |\phi|$

$P.M = -180 - |-142| \rightarrow \boxed{P.M = -38^\circ}$

Exercise 117 - continued

$$L(s) = \frac{k}{s(s+2)(s+10)} \rightarrow \frac{50}{j\omega(j\omega+2)(j\omega+10)}$$

$$L(j\omega) = \frac{50}{j\omega(-\omega^2+12j\omega+20)} \rightarrow L(j\omega) = \frac{50}{(-\omega^3+20\omega)j + 12\omega} \times \frac{12\omega - (-\omega^3+20\omega)j}{(12\omega) - (-\omega^3+20\omega)j}$$

$$L(j\omega) = \underbrace{\frac{50(12\omega)}{(12\omega)^2 - [(-\omega^3+20\omega)j]^2}}_{Re} - j\underbrace{\frac{-\omega^3+20\omega}{(12\omega)^2-[(-\omega^3+20\omega)j]^2}}_{Im}$$

Gain Margin

$$\phi = -180^\circ \Rightarrow Im = 0$$

$$\frac{-(-\omega^3+20)\omega}{(12\omega)^2-[(-\omega^3+20\omega)j]^2} = 0$$

$$\boxed{\omega_f = 4.47 \text{ rad/s}}$$

$$|L(j\omega)| = \frac{50}{4.47\sqrt{4.47^2+4}\sqrt{4.47^2+100}} = 0.208$$

$$G.M = -20\log\left(L(j\omega)_{\omega=\omega_f}\right)$$

$$\boxed{G.M = 13\,dB}$$

(forgot the ⊖ sign in the lecture)

13 dB can be added before instability

Exercise 118

Try this on your own

A unit feedback control system has a loop transfer function

$$L(s) = \frac{k}{(s+1)^2}$$

Determine the gain k so that the phase margin is 60°

Exercise 118 - continued

$K = ?$ for $P.M. = 60^\circ$

$$L(s) = \frac{k}{(s+1)^2} \to \frac{K}{(s+1)(s+1)}$$

at the crossover frequency:

$$\left| \frac{K}{(j\omega_c + 1)^2} \right| = 1$$

$$K = \left(\sqrt{\omega_c^2 + 1^2} \right)^2$$

$$\boxed{\omega_c = \sqrt{K - 1}}$$

$$\varphi = -\operatorname{atan}\left(\frac{\omega_c}{1}\right) - \operatorname{atan}\left(\frac{\omega_c}{1}\right)$$

$$\varphi = -\operatorname{atan}\left(\frac{\sqrt{K-1}}{1}\right) - \operatorname{atan}\left(\frac{\sqrt{K-1}}{1}\right)$$

$$P.M = 180 - |\varphi|$$

$$60^\circ = 180^\circ - 2\operatorname{atan}\left(\sqrt{K-1}\right)$$

$$\boxed{K = 4}$$

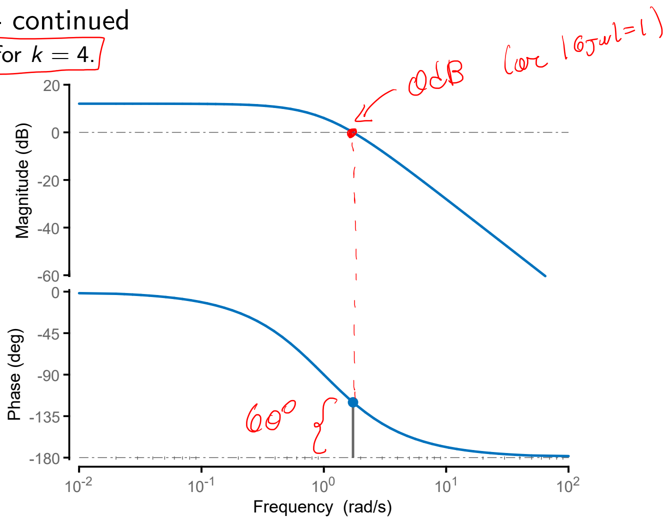
Exercise 118 - continued

$$L(s) = \frac{k}{(s+1)^2}$$

See previous slide.

Exercise 118 - continued

$\boxed{\text{Bode plot for } k = 4.}$



Exercise 119

Home work

A system has a loop transfer function

$$T(s) = 10.5\frac{1 + s/5}{s(1 + s/2)(1 + s/10)}$$

Show that the crossover frequency is 5 rad/s and that the phase margin is 40°

Exercise 119 - continued

$$T(s) = 10.5\frac{1+s/5}{s(\cancel{s}1+s/2)(1+s/10)}$$

$$\left|\frac{10.5\left(1+\frac{j\omega}{5}\right)}{(j\omega)\left(1+\frac{j\omega}{2}\right)\left(1+\frac{j\omega}{10}\right)}\right| = 1 \qquad \frac{10.5\sqrt{1+\frac{\omega^2}{25}}}{\omega\sqrt{1+\frac{\omega^2}{4}}\sqrt{1+\frac{\omega^2}{100}}} = 1$$

$$(10.5)^2\left(1+\frac{\omega^2}{25}\right) = \omega^2\left(1+\frac{\omega^2}{4}\right)\left(1+\frac{\omega^2}{100}\right)$$

solving for $\omega \rightarrow$ $\boxed{\omega = 4.956 \text{ rad/s} \simeq 5}$

Exercise 119 - continued

$$T(s) = 10.5 \frac{1 + s/5}{s(\overset{1}{\cancel{s}} + s/2)(1 + s/10)}$$

$$\phi = \operatorname{atan}\left(\frac{\omega/5}{1}\right) - 90^\circ - \operatorname{atan}\left(\frac{\omega/2}{1}\right) - \operatorname{atan}\left(\frac{\omega/10}{1}\right)$$

$\omega = 5$ rad/s

solve for ϕ

$\boxed{\phi = -139^\circ}$ (at 0dB)

$P.M = 180 - |-139^\circ|$

$\boxed{P.M \simeq 40 \text{ dB}}$

Exercise 120 - continued

Consider a unit feedback system with the loop transfer function

$$L(s) = \frac{k}{s(s+1)(s+4)}$$

(a) For $k = 5$, show that the gain margin is 12 dB

(b) If we wish to achieve a gain margin of 20 dB, determine the value of k

Exercise 120 - continued

(a)

$$L(s) = \frac{k = 5}{s(s+1)(s+4)}$$

$$L(j\omega) = \frac{5}{j\omega(j\omega+1)(j\omega+4)} \rightarrow \frac{5}{-\omega^3 j - 5\omega^2 + 4j\omega} \times \frac{-5\omega^2 - (-\omega^3+4\omega)j}{-5\omega^2 - (-\omega^3+4\omega)j}$$

$Im = 0 \rightarrow \phi = -180^0$

$\omega_f \rightarrow$ the phase is -180^0

$$\frac{-5(-\omega^3+4\omega)}{(-5\omega^2)^2 - [(-\omega^3+4\omega)j]^2} = 0 \qquad 5(\omega^3 - 4\omega) = 0 \qquad \boxed{\omega_f = 2 \text{ rad/s}}$$

At $\omega = \omega_c$

$$|L(j\omega)| = -20\log\left(\frac{5}{2\sqrt{2^2+1}\sqrt{2^2+4}}\right) \qquad \boxed{|L(j\omega)| = +12 \text{ dB}}$$

! G.M.

Exercise 120 - continued

b

!

$$L(s) = \frac{k}{s(s+1)(s+4)}$$

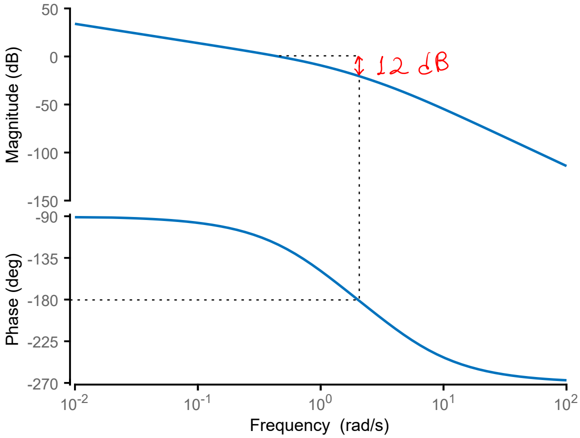
$$20\log\left(\frac{K}{2\sqrt{2^2+1^2}\sqrt{2^2+4^2}}\right) = -20 \text{ dB}$$

$$\frac{K}{2\sqrt{5}\sqrt{20}} = 10^{\frac{-20}{20}}$$

$$\boxed{K=2}$$

Exercise 120 - continued

Bode diagram for $k = 2$.



Exercise 121 - Using Matlab

Consider a unit feedback system with a PI controller such that the loop transfer function is

$$L(s) = \left(k_p + \frac{k_i}{s}\right)\left(\frac{1}{s(s^2 + 3s + 3.5)}\right)$$

with

$$\frac{k_i}{k_p} = 0.2$$

Using Matlab, determine the gain k_p that provides the maximum phase margin. Specify the maximum margin.

Next class...

- Space state models