MECE 3350U
Control Systems

Extra Practice Exercises

Exercise 143

Given the open-loop transfer function

$$G(s) = \frac{s + 0.1}{(s+1)(s+10)(s+100)} \tag{1}$$

(a) Calculate the phase and magnitude of $G(s)$ at $\omega = 10^{-3}$ and 10^{3} rad/sec

(b) Draw the Bode plot

Exercise 143 - continued

$$G(s) = \frac{s+0.1}{(s+1)(s+10)(s+100)} = \frac{j\omega + 0.1}{(j\omega+1)(j\omega+10)(j\omega+100)}$$

$$|G(j\omega)| = \frac{\sqrt{\omega^2+0.1^2}}{\sqrt{\omega^2+1}\sqrt{\omega^2+10^2}\sqrt{\omega^2+100^2}}, \quad \varphi = \operatorname{atan}\left(\frac{\omega}{0.1}\right) - \operatorname{atan}\left(\frac{\omega}{1}\right) - \operatorname{atan}\left(\frac{\omega}{10}\right) - \operatorname{atan}\left(\frac{\omega}{100}\right)$$

	$\lvert G(j\omega)\rvert$	$20\log(\lvert G(j\omega)\rvert)$	φ
$\omega = 10^{-3}$ rad/s	10^{-4}	-80 dB	0.5°
$\omega = 10^{3}$ rad/s	10^{-6}	-120 dB	-173°

Exercise 143 - continued

$$G(s) = \frac{s + 0.1}{(s+1)(s+10)(s+100)}$$

$$G(j\omega) = \frac{0.1\left(\frac{s}{0.1}+1\right)}{(s+1)\,10\left(\frac{s}{10}+1\right)100\left(\frac{s}{100}+1\right)}$$

$$G(j\omega) = \frac{10^{-4}\left(\frac{s}{0.1}+1\right)}{(s+1)\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}$$

$$20\log(10^{-4}) = -80 \text{ dB}$$

cut off frequencies

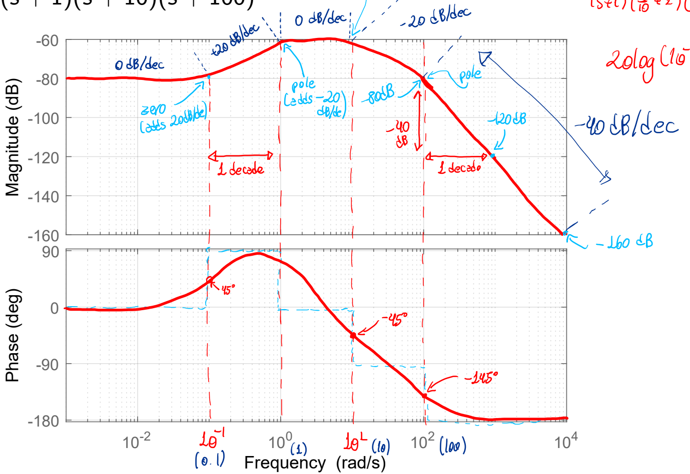
$\omega = 0.1 \rightarrow$ zero +20dB/dec

$\omega = 1$

$\omega = 10$

$\omega = 100$

poles -20dB/dc each



Exercise 144

Given the open-loop transfer function

$$G(s) = \frac{10}{s(s^2 + 0.1s + 25)} \tag{2}$$

(a) Draw the Bode plot

Exercise 144 - continued

$\zeta < 1 \rightarrow$ poles are complex conjugates!

$$G(s) = \frac{10}{s(s^2 + 0.1s + 25)}$$

standard form $\Rightarrow \dfrac{1}{\left(\frac{s}{\omega}\right)^2 + 2\zeta\left(\frac{s}{\omega}\right) + 1}$

$$G(j\omega) = \frac{10}{25s\left(\left[\frac{s}{5}\right]^2 + \frac{0.1}{25}s + 1\right)} \Rightarrow \frac{0.4}{s\left(\left[\frac{s}{5}\right]^2 + 0.02\left(\frac{s}{5}\right) + 1\right)}$$

$\omega_0 = 5$ rad/s

$2\zeta = 0.02$

$\boxed{\zeta = 0.01}$

for complex poles, at the cutoff frequency the gain is

$$-20\log(2\zeta)$$

$$\boxed{= +33\text{ dB}}$$

Exercise 144 - continued

$$G(j\omega) = \frac{0.4}{s\left[\left(\frac{s}{5}\right)^2 + 0.02\left(\frac{s}{5}\right) + 1\right]}$$

(1)

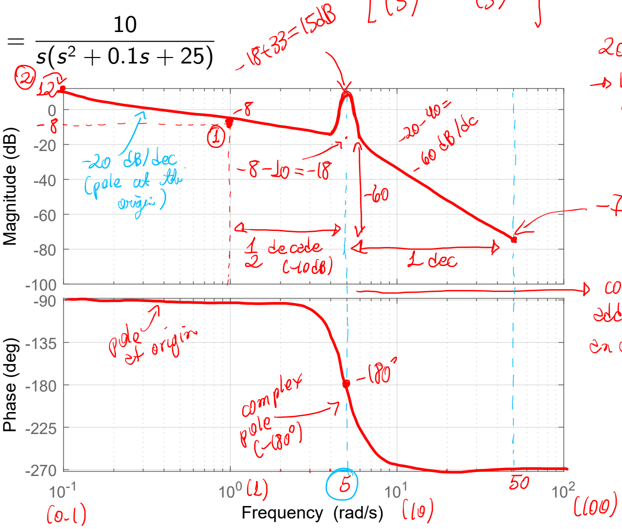
$$G(s) = \frac{10}{s(s^2 + 0.1s + 25)}$$

Start with the pole at the origin $20\log\left(\frac{1}{\omega}\right)$

$\omega = 1$ gives 0, but $20\log(0.4) = -8$

$20\log(0.4) = -8$

→ whole plot shifts down -8 dB



-78dB

→ complex pole adds -40 dB/dec and -180° phase

(2) gain at (2) is

$20\log\left(\frac{1}{\omega}\right) - 8\text{dB}$

$20\log\left(\frac{1}{0.1}\right) - 8\text{dB}$

$= 12\text{ dB}$

Exercise 145

Given the open-loop transfer function

$$G(s) = \frac{20}{(s+1)^2(s+10)} \tag{3}$$

(a) Draw the Bode plot

(b) Calculate the phase and gain margins

(c) Estimate the Nyquist plot and assess stability

(d) Confirm the results with the root-locus

Exercise 145 - continued

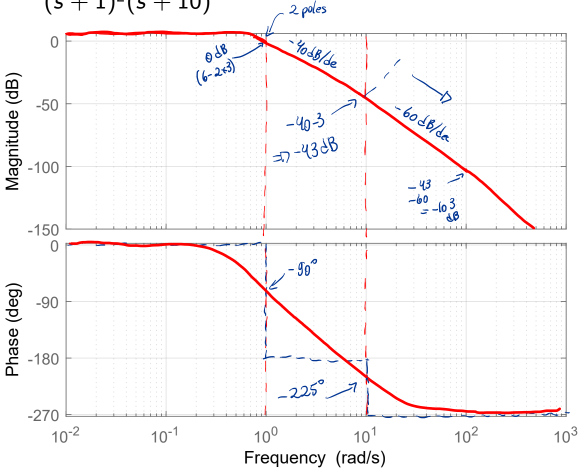
$$\frac{2}{(s+1)^2\left(\frac{s}{10}+1\right)}$$

$20\log(2) = 6dB$

$$G(s) = \frac{20}{(s+1)^2(s+10)}$$

$w = 1 rad/s$

$w = 10 rad/s$



Exercise 145 - continued

$$G(s) = \frac{20}{(s+1)^2(s+10)} \rightarrow \frac{20}{(j\omega+1)^2(j\omega+10)} \Rightarrow \frac{20}{\left(\sqrt{\omega^2+1^2}\right)^2\sqrt{\omega^2+10^2}} = 1$$

$$20^2 = (\omega^2+1)^2(\omega^2+100) \rightarrow \boxed{\omega_c = 1 \text{ rad/s}}$$ (crossover frequency)

phase when $\omega = \omega_c$

$$\varphi = 0 - 2\,\mathrm{atan}\left(\frac{\omega}{1}\right) - \mathrm{atan}\left(\frac{\omega}{10}\right)$$

$$\boxed{\varphi(\omega=\omega_c) = -85^\circ}$$

$$P.M. = 180 - |\varphi|$$

$$\boxed{P.M. = 85^\circ}$$

Exercise 145 - continued

$$\begin{cases} j = \sqrt{-1} \\ j^2 = -1 \end{cases}$$

Gain margin

$$G(s) = \frac{20}{(s+1)^2(s+10)} \Rightarrow \frac{20}{(j\omega+1)^2(j\omega+10)} \rightarrow \frac{20}{-\omega^3 j - 10\omega^2 - 2\omega^2 + 20j\omega + j\omega + 10}$$

$$G(j\omega) = \frac{20}{(10-12\omega^2)+j(-\omega^3+21\omega)} \times \frac{(10-12\omega^2)-j(-\omega^3+21\omega)}{(10-12\omega^2-j(-\omega^3+21\omega)}$$

$$\begin{cases} \text{when } \phi = -180^\circ \\ Im = 0 \end{cases} \Bigg| \text{ what is } \omega_f\,?$$

Im part is $\dfrac{-20(-\omega^3+21\omega)}{(10-12\omega^2)^2 - [j(\omega^3+21\omega)]^2} = 0$

$$-20(-\omega^3+21\omega) = 0$$

$\omega = 0$ ✗ Not valid

$$\boxed{\omega_f = 4.5 \text{ rad/s}}$$

$$|G(j\omega)| = \frac{20}{(\omega^2+1)\sqrt{\omega^2+10^2}}\Bigg|_{\omega=\omega_f}$$

$$|G(j\omega)| = 0.08$$

$$G.M. = -20\log(|G(j\omega)|)$$

$$\boxed{G.M. = 21\text{dB}}$$

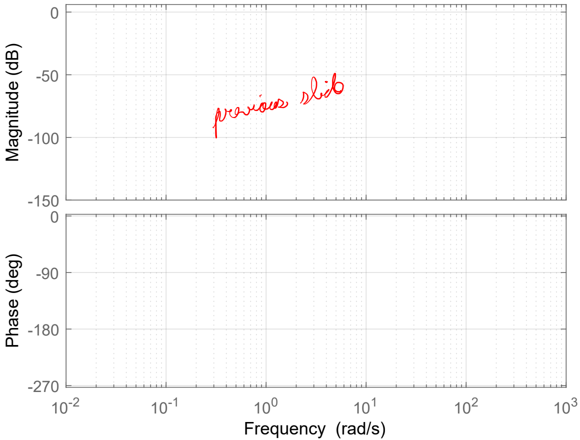
Exercise 145 - continued

$$G(s) = \frac{20}{(s+1)^2(s+10)}$$

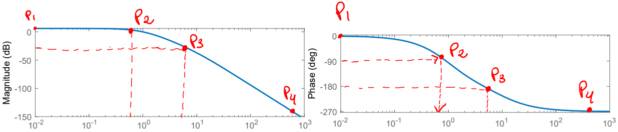
previous slide

Exercise 145 - continued

$$G(s) = \frac{20}{(s+1)^2(s+10)}$$



Exercise 145 - continued

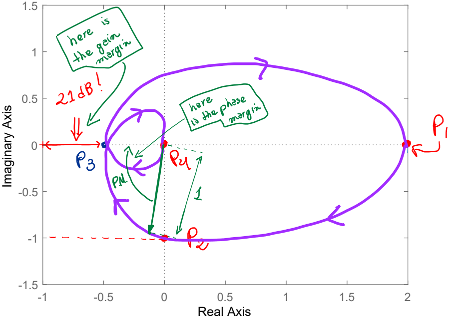


P1 $\begin{cases} \varphi = 0^\circ \\ G = 6\,dB = 2 \end{cases}$

P2 $\begin{cases} \varphi \simeq -90^\circ \\ G \simeq 0\,dB \simeq 1 \end{cases}$

P3 $\begin{cases} \varphi = -180^\circ \\ \boxed{G < 0\,dB < 1} \end{cases}$

P4 $\begin{cases} \omega \to \infty \\ \varphi \to -270^\circ \\ G \to -\infty\,dB = 0 \end{cases}$



$P = 0$

currently $N = 0$

thus $Z = P + N$

$Z = 0$, stable!

if we add 21 dB of gain

$N = +2$

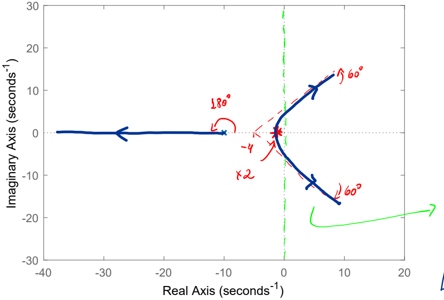
$Z = 0 + 2 = 2$

$\Rightarrow$ two unstable poles!

Exercise 145 - continued

$\alpha = \frac{-1-1-10}{3} = -4$

$\theta = -60^\circ, 60^\circ, 180^\circ$



Two unstable poles after a certain gain (21 dB)

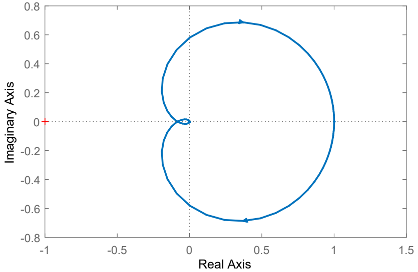
↳ consistent with the Nyquist plot !! (lovely)

Exercise 146

Given the open-loop transfer function and the corresponding Nyquist plot for $k = 1$, determine the maximum gain k before instability.

$$G(s) = k\frac{100}{(s+1)^2(s+10)^2} \tag{4}$$

Homework



Exercise 146

$$G(s) = k\frac{100}{(s+1)^2(s+10)^2}$$

$\Rightarrow \dfrac{100K}{(jw+1)^2(jw+10)^2} = \dfrac{100K}{(w^4-142w^2+100) + j(220w-22w^3)}$

when $\phi = -180^\circ$, the plot crosses the negative real axis, thus $Im = 0$.

$Im = 0$

$100K(220w - 22w^3) = 0$

$w_f = 0$ ✗

$\boxed{w_f = 3.16 \text{ rad/s}}$ ✓

$$|G(jw)| = \frac{100K}{\left(\sqrt{(w^2+1)}\right)^2\left(\sqrt{w^2+10^2}\right)^2}$$

$$\boxed{|G(jw)|_{w=w_f} = 0.0828K}$$

$\Rightarrow \phi = -180^\circ$ thus this is where the Nyquist plot intersects the real axis.

for instability $0.0828K = 1$

$\boxed{K = 12}$

Exercise 146

$$G(s) = k\frac{100}{(s+1)^2(s+10)^2}$$

see previous slide