MECE 3350U
Control Systems

Lecture 7
Steady-State Error

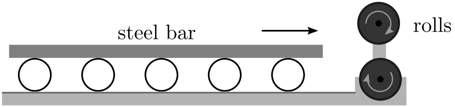
Outline of Lecture 7

By the end of today's lecture you should be able to

- Understand the concept of error and disturbance signals
- Calculate the steady state error of a system
- Analyse the influence of control loop gains

Applications

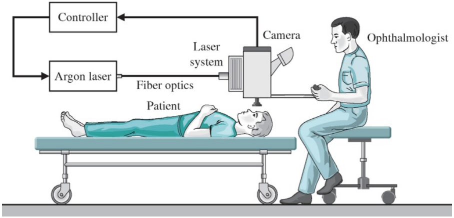
When the steel bars engage in the rolls of the rolling mill, the load on the rolls increases immediately. How can the speed of the rollers be controlled to minimize this disturbance?



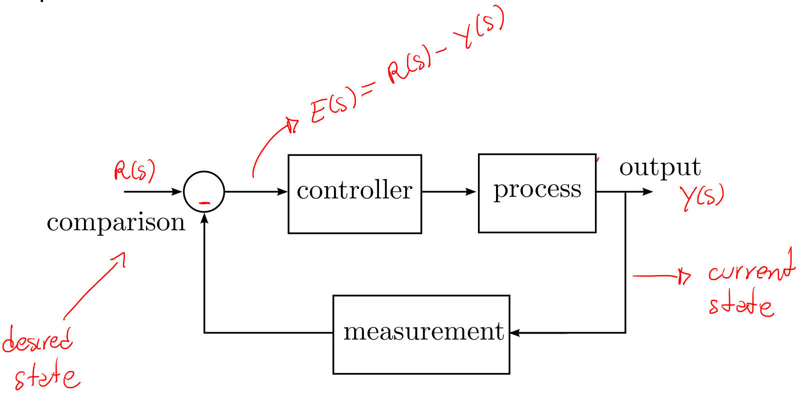
Applications

Automated control of the laser position during eye surgery enables the ophthalmologist to indicate to the controller where lesions should be inserted.

How can we design a controller that minimizes the transient response of the positioning system if the retina moves during the surgery?

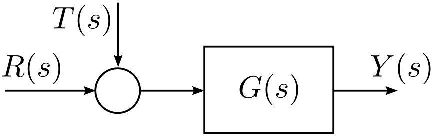


Closed-loop control



Open-loop control

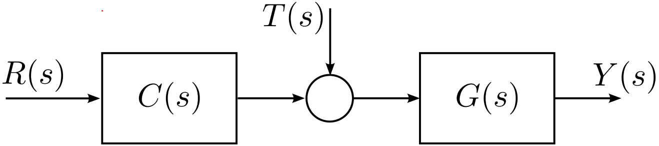
In the absence of a controller and without feedback, the disturbance $T(s)$ directly influences the output $Y(s)$.



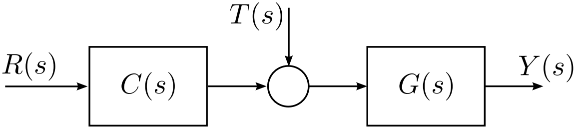
$Y(s) = (R+T)\ G(s)$

unmodelled.

An **open-loop** system operates without feedback and directly generates the output in response to an input signal.

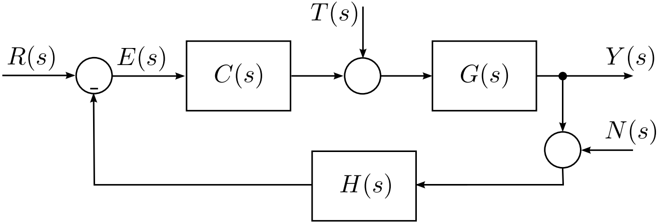


Closed-loop control

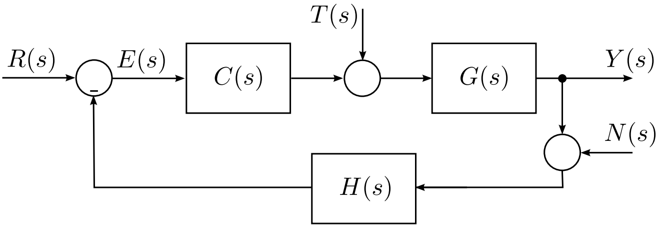


A **closed-loop** system compares the output $Y(s)$ with a desired value $R(s)$.

The error signal $E(s)$ is used by the controller to adjust the actuator.



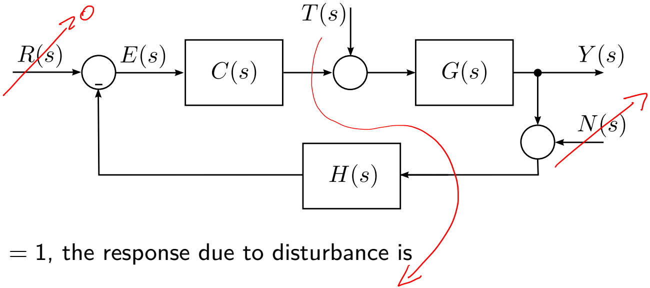
Advantages of closed-loop control



- Decrease sensitivity to variations in the parameters of the process ($G(s)$)
- Improve rejection of the disturbances ($T(s)$)
- Improve noise attenuation ($N(s)$)
- Reduce the steady-state error ($E(s)$)
- Allow for control of the transient response

Disturbance

Disturbance is a change in the values of the nominal parameters of a control system due to external sources.

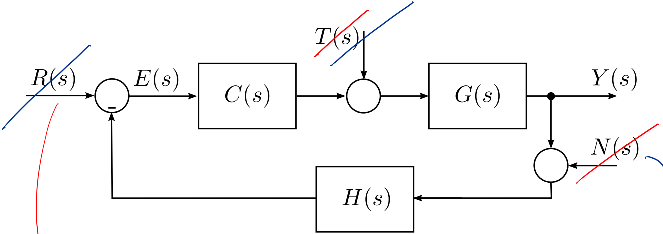


For $H(s) = 1$, the response due to disturbance is

$$Y_D(s) = \frac{G(s)}{1 + C(s)G(s)} T(s)$$

For $R(s) = 0$ and $N(s) = 0$.

Disturbance



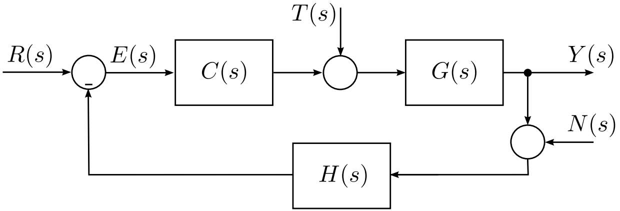
For $H(s) = 1,\ T(s) = 0$; and $N(s) = 0$, the output due to $R(s)$ is

$$Y_R(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} R(s)$$

For $H(s) = 1,\ T(s) = 0$, and $R(s) = 0$, the response to noise is

$$Y_N(s) = -\frac{C(s)G(s)}{1 + C(s)G(s)} N(s)$$

Disturbance and superposition

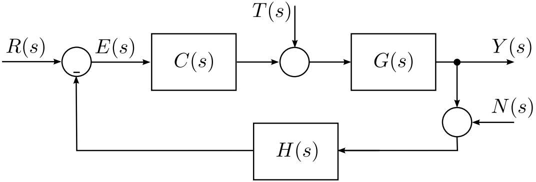


The principle of superposition gives the total response as

$$Y(s) = Y_R(s) + Y_N(s) + Y_T(s)$$

$$Y(s) = \underbrace{\frac{C(s)G(s)}{1 + C(s)G(s)}}_{\text{due to input}} R(s) + \underbrace{\frac{G(s)}{1 + C(s)G(s)}}_{\text{due to disturbance}} T(s) - \underbrace{\frac{C(s)G(s)}{1 + C(s)G(s)}}_{\text{due to noise}} N(s)$$

Disturbance and superposition



Defining the error as $\boxed{E(s) = R(s) - Y(s)}$ gives

$$E(s) = R(s) - \frac{C(s)G(s)}{1 + C(s)G(s)}R(s) - \frac{G(s)}{1 + C(s)G(s)}T(s) + \frac{C(s)G(s)}{1 + C(s)G(s)}N(s)$$

$$E(s) = \left(1 - \frac{C(s)G(s)}{1 + C(s)G(s)}\right)R(s) - \frac{G(s)}{1 + C(s)G(s)}T(s) + \frac{C(s)G(s)}{1 + C(s)G(s)}N(s)$$

$$\boxed{E(s) = \frac{1}{1 + C(s)G(s)}R(s) - \frac{G(s)}{1 + C(s)G(s)}T(s) + \frac{C(s)G(s)}{1 + C(s)G(s)}N(s)}$$

Poles and zeros

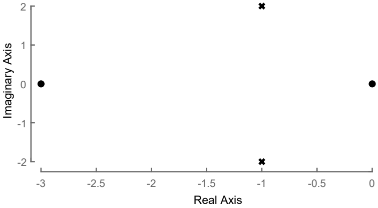
Consider the following function:

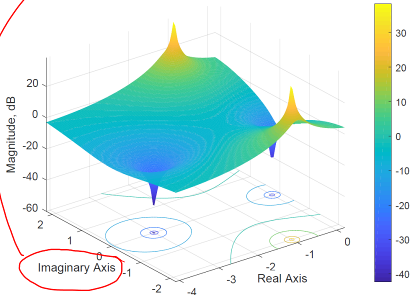
$$F(s) = \frac{s(s+3)}{s^2 + 2s + 5}$$

magnitude changes with frequency!

→ Poles: $-1 + 2j, \; -1 - 2j$

→ Zeros: $0, \; -3$





Disturbance rejection

to minimize $T(s)$

$|C(s)| \rightarrow \infty$

To minimize $N(s)$

$|C(s)| \rightarrow 0$

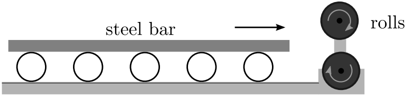
$$E(s) = \frac{1}{1 + C(s)G(s)}R(s) - \frac{G(s)}{1 + C(s)G(s)}T(s) + \frac{C(s)G(s)}{1 + C(s)G(s)}N(s)$$

To reduce the influence of the disturbance:

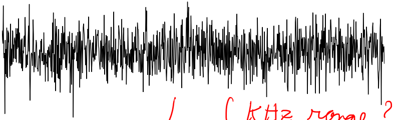
$\rightarrow$ $C(s)$ must be large to minimize the influence of $T(s)$

$\rightarrow$ $C(s)$ must be small to minimize the influence of $N(s)$

How to solve this conflict?



measurement noise

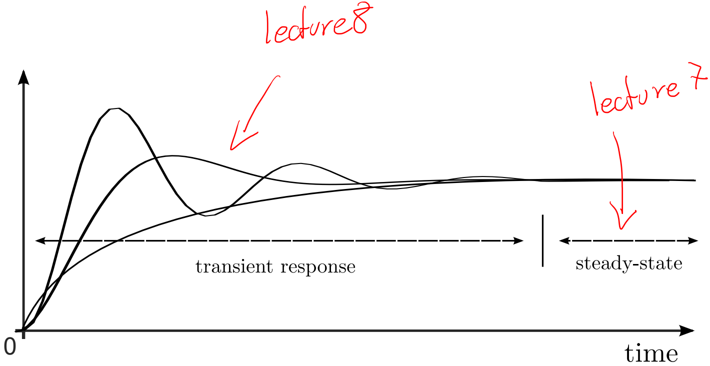


(kHz range?)

$\rightarrow$ Make $C(s)$ large at low frequencies

$\rightarrow$ Make $C(s)$ small at high frequencies

Transient and steady-state response



Steady-state error

controller process input

If $N(s) = T(s) = 0$, the error is:

$$\boxed{E(s) = \frac{1}{1 + C(s)G(s)} R(s)}$$

1

The steady state error can be obtained using the final value theorem:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

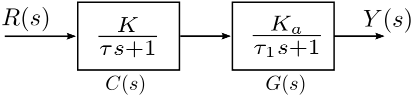
For the closed loop system (step input):

$$e_c(\infty) = \lim s \left(\frac{1}{1 + C(s)G(s)} \right) \left(\frac{1}{s} \right) = \left(\frac{1}{1 + C(0)G(0)} \right)$$

$C(0)G(0)$ is called the "DC gain".

Open-loop *vs* closed-loop

Open-loop: The error is $E(s) = R(s) - Y(s)$



$$E(s) = R(s) - \frac{KK_a}{(\tau s + 1)(\tau_1 s + 1)} R(s)$$

What is the steady-state error?

for $R(s) = \frac{1}{s}$

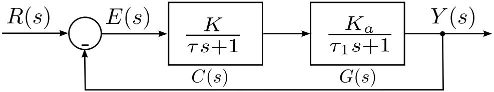
$$e_{ss} = \lim_{s \to 0} E(s).s \Rightarrow \lim_{s \to 0} s \left(\frac{1}{s} - \frac{KK_a}{(\tau s+1)(\tau_2 s+1)} \right) = 1 - KK_a$$

how to make $e_{ss} = 0$?

$K \equiv \frac{1}{K_a}$ → requires a perfect model and no disturbances are allowed

Open-loop *vs* closed-loop

Closed-loop: The error is $E(s) = R(s) - Y(s)$



$$E(s) = R(s) - \frac{KK_a}{(\tau s+1)(\tau_1 s+1)+KK_a}R(s)$$

What is the steady-state error? for $V(s) = \frac{1}{s}$

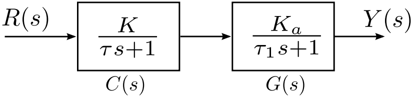
$$\lim_{s \to 0} s\,E(s) \Rightarrow \lim_{s \to 0} \cancel{s}\left(\frac{1}{\cancel{s}} - \frac{1}{\cancel{s}}\,\frac{KKa}{(\tau s+1)(\tau_1 s+1)+KKa}\right)$$

$$= 1 - \frac{KKa}{1+KKa} = \frac{1}{1+KKa}$$

→ error is never zero!

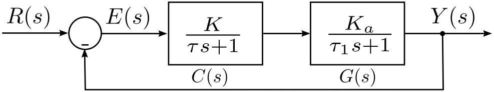
→ can be made small if $KaK \gg 1$

Open-loop *vs* closed-loop



Open-loop: The steady error is

$$E(s) = 1 - KK_a$$



Closed-loop: The steady error is

$$E(s) = \frac{1}{1 + KK_a}$$

Exercise 30

A robotic arm and camera are used to pick fruit as shown in the figure. The camera is used to close the feedback loop to a micro-controller, which controls the arm. The transfer function of the process is

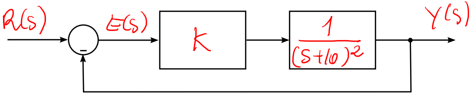
$$G(s) = \frac{1}{(s+10)^2}$$

and the controller is a proportional gain so that $C(s) = K$. Calculate the steady-state error of the gripper for a step command A as a function of K.

$r(s) = \frac{A}{s}$

Exercise 30 - continued

$$G(s) = \frac{1}{(s+10)^2}, \qquad C(s) = K, \qquad R(s) = \frac{A}{s}$$



$$Y(s) = \frac{K}{(s+10)^2+K} R(s) \quad , \quad E(s) = R(s) - Y(s)$$

$$E(s) = \frac{A}{s} - \frac{K}{(s+10)^2+K} \cdot \frac{A}{s} \quad , \quad e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} \cancel{s}\left(\frac{A}{\cancel{s}} - \frac{K}{(s+10)^2+K} \frac{A}{\cancel{s}}\right)$$

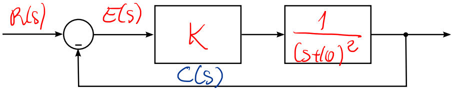
$$e_{ss} = A - \frac{AK}{100+K} = A\frac{100+K-K}{100+K} = \frac{100}{100+K}A //$$

Exercise 30 - continued

Alternative solution

$$E(s) = \left(\frac{A}{s}\right)\frac{1}{1 + K/(s+10)^2}$$

Eq. (1), slide 26



$$E(s) = \frac{R(s)}{1 + G(s)C(s)} \rightarrow E(s) = \frac{A}{s}\left(\frac{1}{1 + \frac{K}{(s+10)^2}}\right)$$

$$e_{ss} = \lim_{s \to 0} E(s) \cdot s = \lim_{s \to 0} s \cdot \frac{A}{s}\left(\frac{1}{1 + \frac{K}{(s+10)^2}}\right) = \frac{A}{1 + \frac{K}{100}}$$

$$= \frac{100}{100 + K} A \;//$$

Exercise 31

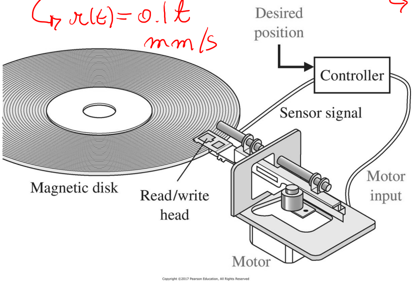
A hard drive requires a motor to position a read/write head over the tracks of data on a spinning disk. The motor and head have the transfer function

$$G(s) = \frac{100}{s(0.001s + 1)}$$

The controller is $C(s) = K$. Calculate K that yields a steady-state error of 0.1 mm for a ramp input of 10 cm/s.

$\to r(t) = 0.1t$ mm/s

$\to e_{ss} = \lim_{s \to 0} sE(s) = 0.1$

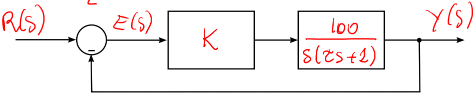


Exercise 31 - continued

$\tau = 0.001$

$r(t) = 0.1t$

$$G(s) = \frac{100}{s(\underbrace{0.001}_{\tau}s+1)}, \qquad C(s) = K, \qquad R(s) = \frac{0.1}{s^2}$$



$$Y(s) = \frac{100K}{s(s\tau+1)+100K}R(s)$$

$$E(s) = R(s) - Y(s) = \frac{0.1}{s^2}\left(1 - \frac{100K}{s(s\tau+1)+100K}\right)$$

$$E(s) = \frac{0.1}{s^2}\left(\frac{s(s\tau+1)+100K - 100K}{s(s\tau+1)+100K}\right)$$

$$E(s) = \frac{0.1}{s^2}\frac{s(s\tau+1)}{s(s\tau+1)+100K} \rightarrow E(s) = \frac{0.1}{s}\frac{s\tau+1}{(s)(s\tau+1)+100K}$$

Exercise 31 - continued

$$E(s) = \frac{0.1}{s} \frac{s\tau + 1}{(s)(s\tau+1) + 100K}$$

$$e_{ss} = \lim_{s \to 0} E(s).s \rightarrow \lim_{s \to 0} \cancel{s}\left(\frac{0.1}{\cancel{s}} \frac{s\tau+1}{s(s\tau+1)+100K}\right) = \frac{1}{100K} \quad ''$$

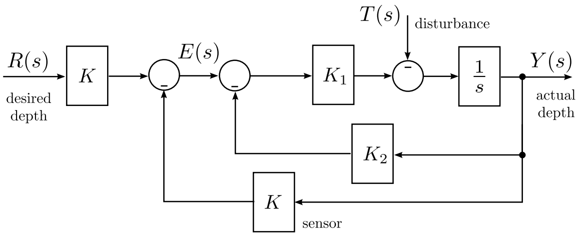
from the problem, we want $e_{ss} = 0.1\,mm$ or $\frac{0.1}{1000}\,m$

thus: $\frac{1}{100K} = \frac{0.1}{1000}$, solving for K

$$\boxed{K = 100}$$

Exercise 32

A submarine has a depth control system as illustrated.



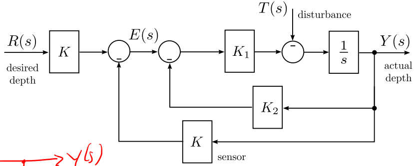
(a) Determine the transfer function $F(s) = Y(s)/R(s)$

(b) Determine the steady state error for $T(s) = 1/s$ and $R(s) = 0$

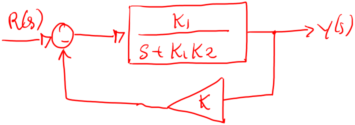
(c) Calculate the response $y(t)$ for $R(s) = 1/s$ when $K = K_2 = 1$ and $1 < K_1 < 10$. Select K_1 for the fastest response.

Exercise 32 - continued

(a) The transfer function $Y(s)/R(s)$



Set $T(s)=0$



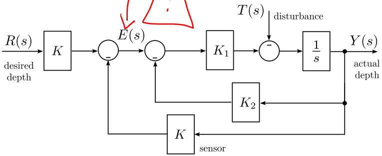
$$Y(s) = \frac{KK_1}{s + K_1K_2 + KK_1} R(s),$$

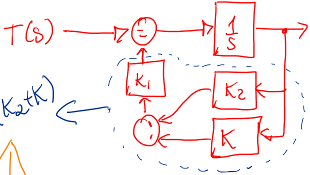
$$\boxed{Y(s) = \frac{KK_1}{s + K_1(K + K_2)} R(s)}$$

Exercise 32 - continued

(*b*) The state state error for $T(s) = 1/s$

set $R(S) = 0$





$K_1(K_2+K)$

$$Y(s) = -\frac{1}{S+K_1(K_2+K)}\,T(s)$$

$$E(s) = \cancel{R(s)} - K\,Y(s)$$

0

$$E(s) = -K\,Y(s) = \frac{K}{S+K_1(K_2+K)}\cdot\frac{1}{S}$$

$[T(s)]$

$$\lim_{s\to 0} sE(s) = \frac{K}{K_1(K_2+K)} \quad //$$

Exercise 32 - continued

(c) $y(t)$ for $R(s) = 1/s$, $\boxed{K = K_2 = 1}$ and $1 < K_1 < 10$ for the fastest response.

from (a)

$$Y(s) = \frac{KK_1}{s + K_1(K + K_2)} R(s) =$$

replacing $K = K_2 = 1$, $R(s) = \frac{1}{s}$

$$Y(s) = \frac{K}{s + 2K_1} \cdot \frac{1}{s}$$

partial fraction $\rightarrow$ $Y(s) = \frac{1}{2}s - \frac{1}{2}\frac{1}{s+2K_1}$

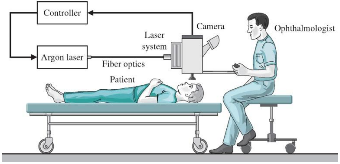
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\left(1 - e^{-2K_1 t}\right)$$

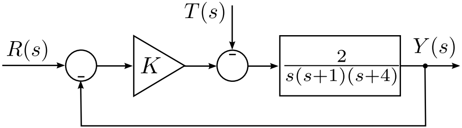
for the fastest response, select $K_1 = 10$

Exercise 33 - Design problem

See Matlab code on Blackboard.

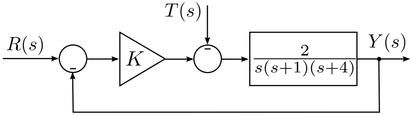
The position control system of a laser system for ophthalmologic surgery is shown in the figure. Select the gain K to ensure an appropriate transient response to a step change in $R(s)$ and the effect of disturbance is minimized. The steady-state error must be zero. The system is stable provided that $K < 10$. Plot the step response of the system for $1 < K < 15$ using Matlab.





Exercise 33 - continued

Set $T(s)=0$



$$\frac{Y(s)}{R(s)} = \frac{2K}{s(s+1)(s+4)+2K}$$

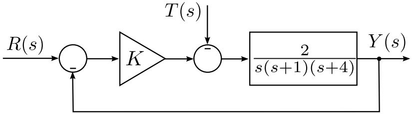
$$E(s) = R(s) - Y(s)$$

$$E(s) = \frac{1}{s}\left(1 - \frac{2K}{s(s+1)(s+4)+2K}\right)$$

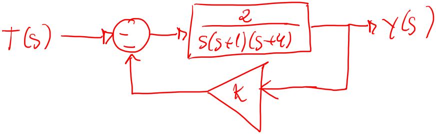
$$e_{ss} = \lim_{s \to 0} s\,E(s)$$

$$\boxed{e_{ss} = 0 \quad \forall \; K}$$

Exercise 33 - continued



Select gain K with $1 \leq K \leq 10$ to minimize $Y(s)$ when $\begin{cases} T(s) \neq 0 \\ R(s) = 0 \end{cases}$



$$Y(s) = -\frac{2}{s(s+1)(s+4)+2K} T(s), \quad T(s) = \frac{1}{s}$$

$$y_{ss} = \lim_{s \to 0} s\,Y(s) = -\frac{A}{K} \Rightarrow \text{select } K \text{ that minimize } y_{ss},\ K=10$$

See Matlab code posted on B.B.

→ plot $y(t)$ for different K.

→ Note that:

- $K<10$ → stable
- $K=10$ → oscillatory
- $K>10$ → unstable

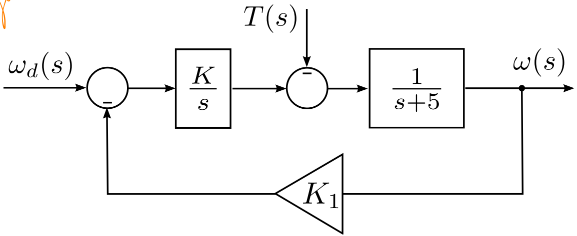
→ Note how the location of the poles change with K

Exercise 34 - Design problem

Consider the speed control system shown.

Homework

try this on your own

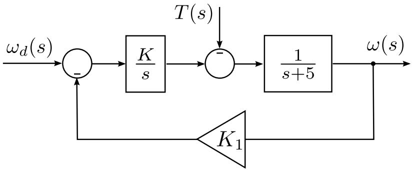


Determine:

(a) The range of K_1 so that the steady state tracking error is $e \leq 1\%$

(b) KK_1 so that the steady state error for $T(t) = 2t$ mrad/s for $0 \leq t \leq 5$ sec is less than 0.1 mrad/s.

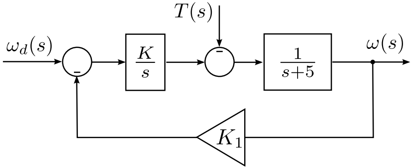
Exercise 34 - continued



Answer: ⓐ $0.99 < K_1 < 1.01$

ⓑ $KK_1 > 20$

Exercise 34 - continued



Next episode...

- Transient performance