MECE 3350U
Control Systems

Lecture 12
The Root-Locus Method 2/2

Videos in this lecture

Lecture: https://youtu.be/73Tga4R4nP8

Exercise 60: https://youtu.be/ylKnzGp84nc

Exercise 61: https://youtu.be/1soTbFnY-RY

Exercise 62: https://youtu.be/W3ePdb7k1-Q

Exercise 63: https://youtu.be/gWaHo3xYSzg

Exercise 64: https://youtu.be/n2BwSae7UQo

Exercise 65: https://youtu.be/41Z8xNG9-Mg

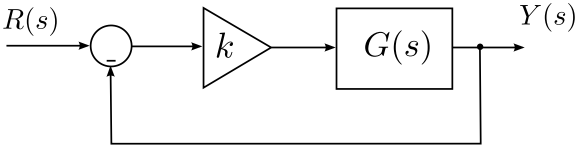
Outline of Lecture 12

By the end of today's lecture you should be able to

- Extend the concept of root locus
- Understand the applications of the Root-Locus method
- Apply Root-Locus method to a design problem

Review

Consider the following closed-loop system:



The closed-loop transfer function is

$$T(s) = \frac{kG(s)}{1 + kG(s)} \tag{1}$$

And the characteristic equations is

$$1 + kG(s) = 0 \tag{2}$$

Review

To analyse the influence of a given parameter of interest k, the characteristic equation of the closed-loop system must in the format

$$1 + kH(s) = 0 \tag{3}$$

$\Rightarrow$ k is the parameter of interest

$\Rightarrow$ $H(s)$ is a function of s

$$1 + k\frac{P(s)}{Q(s)} = 0 \tag{4}$$

If $k = 0$, the poles of $T(s)$ are the roots of $Q(s)$

If $k \to \infty$, the poles of $T(s)$ are the roots of $P(s)$

The root locus is the set of values of s for which $1 + kH(s) = 0$ is satisfied as the real parameter k varies from 0 to ∞.

Angle requirement

$$1 + kG(s) = 0, \;\; kG(s) = -1 + j0 \tag{5}$$

If the open loop transfer function is

$$G(s) = k\frac{(s+z_1)(s+z_2)(s+z_3)\dots(s+z_m)}{(s+p_1)(s+p_2)(s+p_3)\dots(s+p_n)} \tag{6}$$

The magnitude requirement for root locus is

$$|G(s)| = k\frac{|s+z_1||s+z_2||s+z_3|\dots|s+z_m|}{|s+p_1||s+p_2||s+p_3|\dots|s+p_n|} = 1 \tag{7}$$

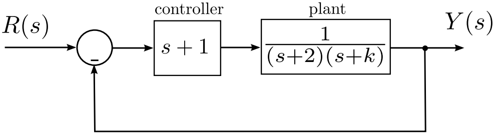
The angle requirement for root locus is

$$\begin{aligned}\angle G(s) = \quad & \angle(s+z_1) + \angle(s+z_2) + \dots \\ & - [\, \angle(s+p_1) + \angle(s+p_2) + \dots \,] = 180^\circ + \ell 360^\circ\end{aligned}$$

where $\ell = 1, 2, 3$ ...

Example 1

Sketch the root locus of the closed loop system as k varies from 0 to ∞.



Find the closed-loop transfer function.

$$T(s) = \frac{\frac{(s+1)}{(s+2)(s+k)}}{1 + \frac{(s+1)}{(s+2)(s+k)}}$$

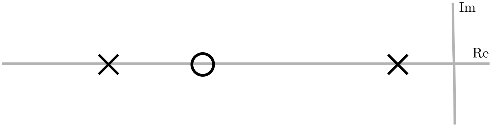
Prepare the characteristic equation as $1 + kH(s)$.

Example 1 - continued

$$1 + k\frac{s+2}{s^2+3s+1} = 1 + k\frac{s+2}{(s+2.618)(s+0.382)}$$

Locate the poles and zeros of $H(s)$ as defined in (2)

Draw the approximate root locus



What can we conclude?

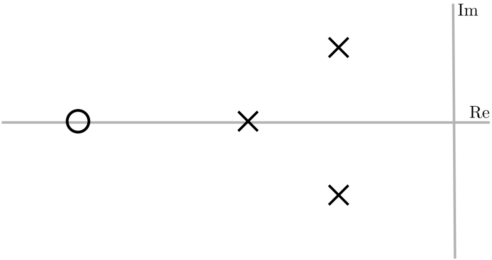
Example 2

Find the root locus for as system whose characteristic equation is

$$1 + k\frac{s+1}{s^3+s^2+3s+1}.$$

The above can be rewritten as

$$1 + k\frac{s+1}{(s+0.36)(s+0.32+j1.63)(s+0.32-j1.63)}.$$



Breakaway point

For the characteristic equation

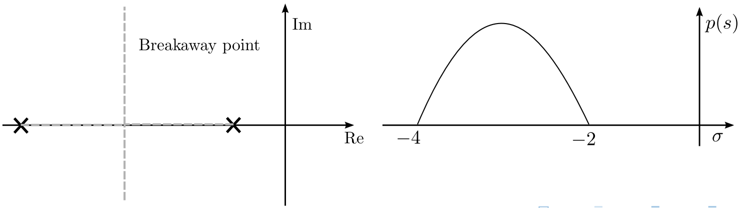
$$1 + k\frac{Q(s)}{P(s)} = 0 \tag{8}$$

Let $k = p(s)$ and rearrange (8) so that

$$p(s) = -\frac{P(s)}{Q(s)} \tag{9}$$

The breakaway point satisfies

$$\frac{dp(s)}{ds} = \frac{dk}{ds} = 0 \tag{10}$$



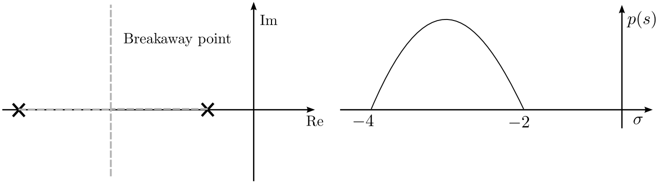
Breakaway point - Example

Find the breakaway point for

$$1 + k\frac{1}{(s+2)(s+4)} = 0. \tag{11}$$

Let $k = p(s)$ and rewrite (11) as in (9)

$$p(s) = -(s+2)(s+4) = \tag{12}$$



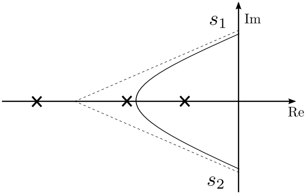
Imaginary axis crossing point

The point where the root locus crosses the imaginary axis indicates the maximum gain k_{max} before instability.

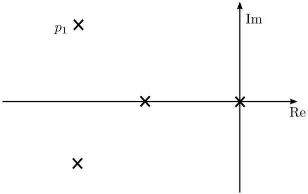
How to find k_{max}? $\Rightarrow$ Use the Routh-Hurwitz method. Example:

$$1 + k\frac{1}{(s+1)(s+2)(s+3)} = 0 \rightarrow s^3 + 6s^2 + 11s + 6 + k = 0$$

s_3	1	11
s_2	6	$6+k$
s_1	$\frac{6+k-66}{-6}$	0
s_0	$6+k$	0



Departure angle from a pole



In what direction does p_1 move?

Recall the angle requirement:

$$\angle G(s) = (s + z_1) + \angle(s + z_2) + \dots$$
$$- [\, \angle(s + p_1) + \angle(s + p_2) + \dots \,] = 180^\circ + \ell 360^\circ$$

$$\sum \psi - \sum \phi = 180^\circ + 360^\circ(\ell)$$

where

→ ψ is the angle from a point to a zero;

→ ϕ is the angle from a point to a pole;

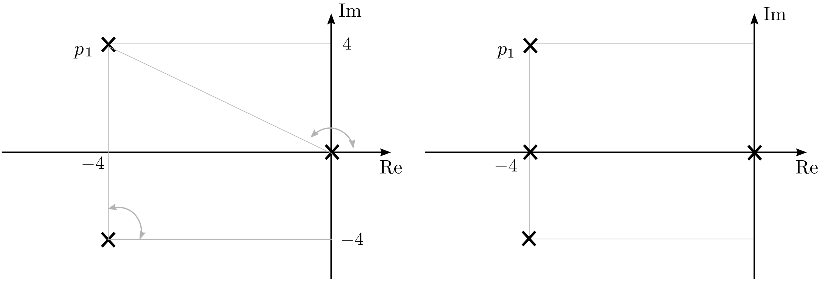
For any point in the plane we can write

$$q\phi = \sum \psi - \sum \phi - 180^\circ - \ell 360^\circ$$

ℓ is an integer.

Departure from a pole

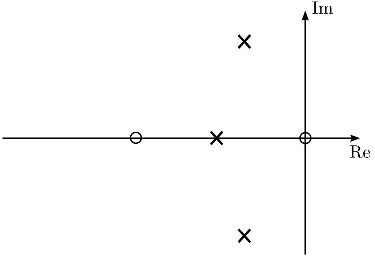
In what direction does p_1 move?



$$q\phi = \sum \psi - \sum \phi - 180^\circ - \ell 360^\circ$$

Arrival at a zero

The same approach can be used to determine the angles of arrival of a branch at a zero.



$$q\psi = \sum \phi - \sum \psi + 180^\circ + \ell 360^\circ$$

The 10 rules for drawing the root-locus

Rule 1: As k varies from 0 to ∞, there are n lines (loci) where n is the degree of $Q(s)$ or $P(s)$, whichever is greater.

Rule 2: As k varies from 0 to ∞, the roots of the characteristic equation move from the poles to the zeros of $H(s)$.

Rule 3: The root loci must be symmetrical with respect to the horizontal axis.

Rule 4: The a root cannot cross its path

Rule 5: The loci are on the real axis to the left of an odd number of poles and zeros that lie on the real axis.

Rule 6: Lines leave (break out) and enter (break in) the real axis at 90°

The 10 rules for drawing the root-locus

Rule 7: If there a different number of poles and zeros, extra lines that do not have a pair go to or come from infinity.

Rule 8: The angle of the asymptotes of the curves that go to infinity is

$$\phi = \frac{180^\circ + 360^\circ(q-1)}{n-m}, \;\; q = 1, 2, \ldots, n-m \tag{13}$$

and they radiate out from the real axis at

$$\alpha = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} \tag{14}$$

Rule 9: If there is a least two lines that go to infinity, then the sum of all the roots is constant.

Rule 10: If the gain sweeps from 0 to $-\infty$, the root loci can be drawn by reversing Rule 5 **and** adding a 180° to the asymptote angles.

Steps for drawing the root locus

Step 1

Prepare the characteristic equation in the form of

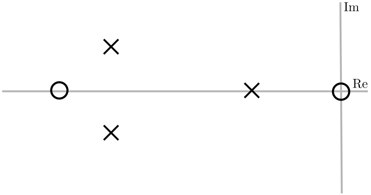
$$1 + kH(s) = 0 \tag{15}$$

and factor the m poles and n zeros

$$1 + k\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} \tag{16}$$

Step 2

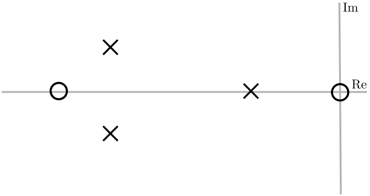
Locate the poles and zeros of $H(s)$ in the plane



Steps for drawing the root locus

Step 3

Locate the segments of the of the real axis that are root loci. Root loci are to the left of an odd number of poles and zeros.



Step 4

Calculate the angle θ and centre α of asymptotes of loci that tend to infinity

$$\theta = \frac{180^\circ + 360^\circ(q-1)}{n-m} \qquad \alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

Steps for drawing the root locus

Step 5

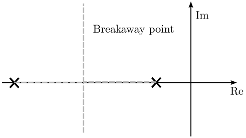
If applicable, determine the points at which the loci cross the imaginary axis. Use Routh-Hurwitz criterion.

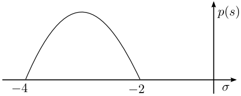
s_3	a_3	a_1
s_2	a_2	a_0
s_1	b_1	0
s_0	c_1	0

Step 6

If applicable, determine the breakaway point on the real axis. Set $p(s) = k$, and find the minimum and/or maximum values of $p(s)$, i.e,:

$$p(s) = -\frac{1}{H(s)}, \rightarrow \frac{d}{ds}\left[-\frac{1}{H(s)}\right] = 0$$





Steps for drawing the root locus

Step 7

Determine the angle of locus departure from complex poles and the angle of locus at arrival at complex zeros using the phase criterion.

Departure from a complex pole:

$$q\phi = \sum \psi - \sum \phi - 180^\circ - \ell 360^\circ$$

Arrival at a complex zero

$$q\psi = \sum \phi - \sum \psi + 180^\circ + \ell 360^\circ$$

Step 8

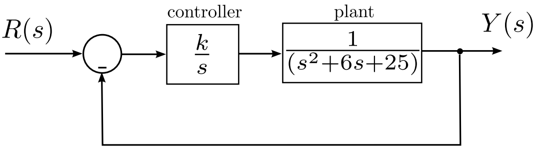
Complete the root locus

Step 9

You may check you results using the Matlab function "rlocus(H);".

Exercise 60

Determine the root-locus plot for the following closed-loop system.

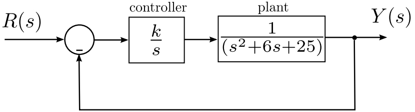


Procedure:

Follow the steps given in this lecture.

Exercise 60 - continued

Step 1: Preparing the characteristic equation.

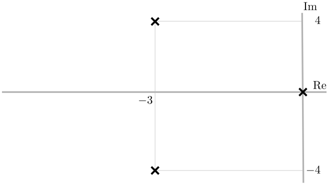


Step 2: Poles and zeros

Step 3: Locate the segments of the loci

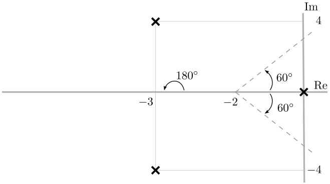
Exercise 60 - continued

Step 4: The angle and centre of asymptotes



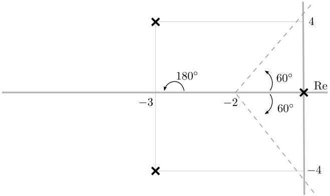
$$\theta = \frac{180^\circ + 360^\circ(q-1)}{n-m} \qquad \alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

Exercise 60 - continued



Step 5: Points where the loci cross the imaginary axis

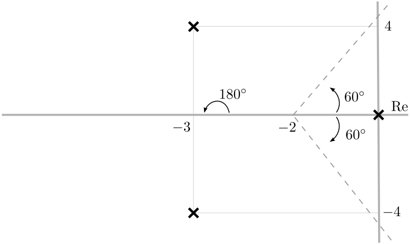
Exercise 60 - continued



Step 6: Breakaway points

$$G(s) = \frac{1}{s(s^2 + 6s + 25)} \tag{17}$$

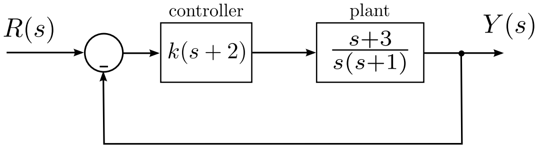
Exercise 60 - continued



Step 7: Angle of departure from the complex poles

Exercise 61

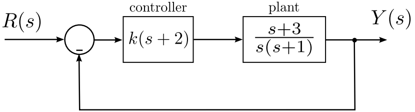
Sketch the root loci for the system shown. Determine the range of k for which the closed-loop system is underdamped.



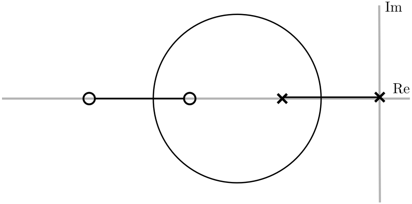
Follow the steps given in this lecture.

Determine the range of k for which the roots are complex.

Exercise 61 - continued



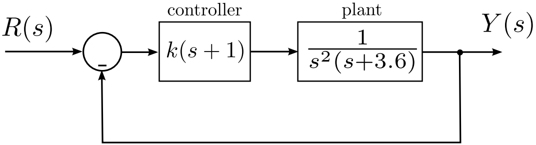
Exercise 61 - continued



$$k = -\frac{s(s+1)}{(s+3)(s+2)}$$

Exercise 62

Sketch the root loci for the system shown.



Follow the steps given in this lecture.

Exercise 62 - continued

$$H(s) = \frac{s+1}{s^2(s+3.6)}$$

Exercise 62 - continued

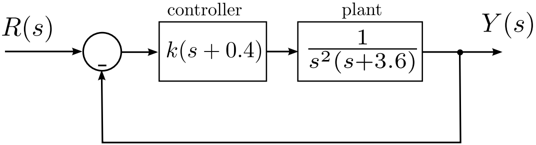
$$H(s) = \frac{s+1}{s^2(s+3.6)}$$

Asymptotes: 90°, -90°, at $s = -1.3$.

Breakaway: $s = 0$

Exercise 63

Sketch the root loci for the system shown.



Follow the steps given in this lecture.

Exercise 63 - continued

$$H(s) = \frac{s + 0.4}{s^2(s + 3.6)}$$

Exercise 63 - continued

$$H(s) = \frac{s + 0.4}{s^2(s + 3.6)}$$

Exercise 63 - continued

$$H(s) = \frac{s + 0.4}{s^2(s + 3.6)}$$

Asymptotes: 90°, -90°, at $s = -1.6$.

Breakaway: $s = 0, s = -1.2$

Exercise 63 - note

An alternative way to find the root locus is to use the angle requirement.

$$\angle\left[k\frac{s+0.4}{s^2(s+3.6)}\right] = 180^\circ + \ell 360^\circ$$

which can be rewritten as

$$\angle(s+0.4) - 2\angle(s) - \angle(s+3.6) = 180^\circ + \ell 360^\circ.$$

Since $s = \sigma + j\omega$

$$\angle(\sigma + j\omega + 0.4) - 2\angle(\sigma + j\omega) - \angle(\sigma + j\omega + 3.6) = 180^\circ + \ell 360^\circ$$

and $\angle s = \tan^{-1}(\omega/\sigma)$, the root locus function satisfies

$$\tan^{-1}\left(\frac{\sigma}{\omega+0.4}\right) - 2\tan^{-1}\left(\frac{\sigma}{\omega}\right) - \tan^{-1}\left(\frac{\sigma}{\omega+3.6}\right) = 180^\circ + \ell 360^\circ$$

The angle of the root locus then is $d\omega/d\sigma$.

Exercise 64

Consider a unit feedback system with an open loop transfer function

$$L(s) = \frac{k}{s^3 + 50s^2 + 500s + 1000}$$

(a) Find the breakaway point on the real axis

(b) Find the asymptote centroid

(c) Find the value of k at the breakaway point

(d) Draw the root locus for $k \geq 0$

Exercise 64 - continued

$$1 + \frac{k}{s^3 + 50s^2 + 500s + 1000} = 0$$

Exercise 65

The primary mirror of a large telescope can gave a diameter of 10 m and a mosaic of 36 hexagonal segments with the orientation of each segment actively controlled. Suppose this unit feedback system for the mirror segments has the open loop transfer function

$$L(s) = k\frac{1}{s(s^2+2s+5)}.$$

(a) Find the asymptotes and sketch them in the s-plane

(b) Find the angle of departure from the complex poles

(c) Determine the gain when 2 roots lie on the imaginary axis

(d) Sketch the root locus

Exercise 65 - continued

$$1 + k\frac{1}{s(s^2+2s+5)} = 0$$

(a) Find the asymptotes and sketch them in the s-plane

(b) Find the angle of departure from the complex poles

Exercise 65 - continued

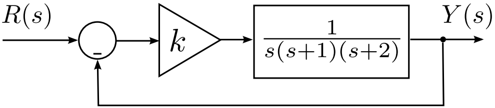
$$1 + k\frac{1}{s(s^2+2s+5)} = 0$$

(c) Determine the gain when 2 roots lie on the imaginary axis

(d) Sketch the root locus

Skills check 34 - From a midterm examination

Recent developments of micro-electromechanical systems made is possible to fabricate untethered biomedical microrobots that can be injected intravenously to accomplish targeted drug delivery tasks. The position $Y(s)$ of such a microrobot navigating in microfluidic arterial bifurcations can be controlled by the action of an external magnetic field $R(s)$. For the system shown:

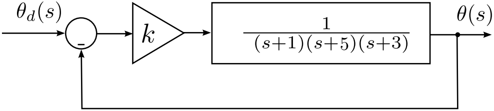


(a) Draw the root-locus of the closed-loop feedback system as a function of k. Specify the asymptote angles and centroid, and the breakaway point, if any.

(b) Based on the root-locus, determine the range of k that results in an over-damped system response, a critically damped response, and an unstable system.

Skills check 35 - From a midterm examination

Vertical takeoff, vertical landing (VTVL) is a form of takeoff and landing for rockets. The guidance system must be capable of measuring the position and altitude of the rocket and small deviations from the vertical can cause large deviations from the intended trajectory. A feedback control system such as the one shown below is required to keep the rocket at the correct angle $\theta_d(s)$.



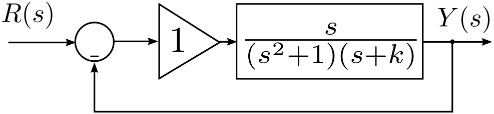
(a) Draw the root-locus of the closed-loop system a k varies from 0 to infinity. Specify the breakaway point, if any, and the centroid of the asymptotes.

(b) Specify the range of k that results in: an overdamped response, a critically damped response, and a under-damped response. The maximum gain before instability is $k = 190$.

(c) Using the root-locus, calculate the natural frequency of the dominant poles when the damping ratio is 0.6.

Skills check 36 - From a final examination

The feedback control system for a vertical takeoff vertical landing system shown below is required to keep the rocket at the correct angle $\theta_d(s)$.



(a) Draw the root-locus of the closed-loop system a k varies from 0 to infinity. Specify the breakaway or breakin point(s), if any.

(b) Specify the range of k that results in: an overdamped response, a critically damped response, and a under-damped response, and an unstable system.

(c) When $k = 1$, determine the location of the poles on the root-locus, the damping ratio, and the natural frequency.

Answer to skills check

S34

$0 < k < 0.38$	over-damped
$k = 0.38$	critically damped
$0.38 < k < 6$	underdamped
$k = 6$	marginally stable
$k > 6$	unstable

S35

$0 < k < 3$	over-damped
$k = 3$	critically damped
$3 < k < 190$	underdamped
$k > 190$	unstable
(c)	$\omega_n = 2.83$ rad/s

Next class...

- PID controllers