MECE 3350U
Control Systems

Lecture 9
Dominant Poles and Zeros

Videos in this lecture

Lecture: https://youtu.be/m7hL8qP1I1c

Exercise 40: https://youtu.be/3HyWc3hegU0

Exercise 41: https://youtu.be/HjDFLWMhszQ

Exercise 42: https://youtu.be/AgWkqeiacx0

Exercise 43: https://youtu.be/cq3NWWulFqc

Exercise 44: https://youtu.be/NGewENVDRww

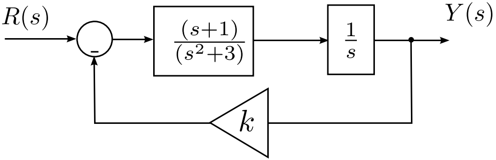
Outline of Lecture 9

By the end of today's lecture you should be able to

- Understand the concept of dominant poles
- Recognize the influence zeros on the transient repose
- Simplify a transfer function to lower orders

Applications

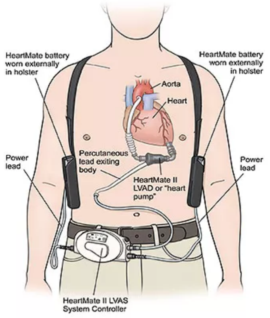
The roll control autopilot of an aircraft has the following structure:



How can we calculate the *k* that yields an overshoot of less than 2%?

Applications

A ventricular assist device is a mechanical pump used to support heart function and blood flow in people with weak or failing hearts.



The model of the heart and pump system results in a third order transfer function. How can we analyse the transient response of the system?

First order system

Consider the response of a first order system to an unit step input:

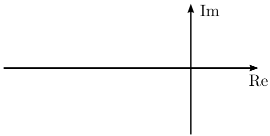
$$X(s) = \frac{1}{s+a}\left(\frac{1}{s}\right)$$

Using partial fraction expansion:

$$X(s) = \frac{1/a}{s} - \frac{1/a}{s+a}$$

The inverse transform yields

$$x(t) = \frac{1}{a}(1 - e^{-at})$$



The transfer function has one pole located at $s = -a$.

→ How does the magnitude of $s = -a$ influence the transient response?

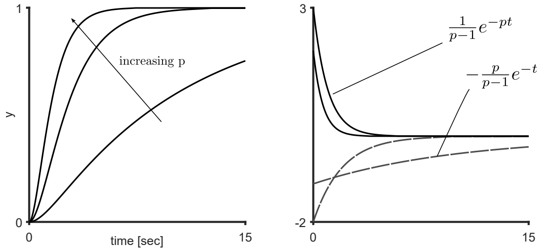
The effect of an additional pole

Let us now examine the step response of

$$X(s) = \frac{p}{(s+1)(s+p)}\left(\frac{1}{s}\right) = \frac{1}{(s+1)(\frac{1}{p}s+1)}\left(\frac{1}{s}\right).$$

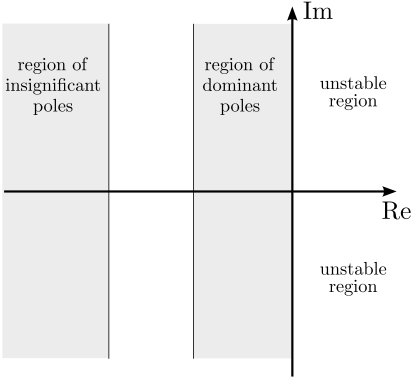
Partial fraction expansion gives:

$$x(t) = 1 - \frac{p}{p-1}e^{-t} + \frac{1}{p-1}e^{-pt}$$



Conclusion: If $p >> 1$, the term $1/(p-1)e^{-pt}$ is negligibly small as $t \rightarrow \infty$.

The effect of an additional pole



If the magnitude of the real part of a pole is at least 5 to 10 times that of a dominant pole, then the pole may be regarded as insignificant.

Second order systems with an additional pole

Consider the 3rd order function

$$T(s) = \frac{1}{(s^2 + 2\zeta\omega_n s + 1)(\gamma s + 1)}.$$

Real part of the poles are: $-1/\gamma$ and $-\zeta\omega_n$. Thus, if

$$\left|\frac{1}{\gamma}\right| \geq 10|\zeta\omega_n| \tag{1}$$

The response can be approximated by

$$T_a(s) = \frac{1}{s^2 + 2\zeta\omega_n s + 1}.$$

Take $\omega_n = 1$, and $\zeta = 0.45$: gives two poles at $s = -0.45 \pm 0.89i$.

Example 1: $\gamma = 1.00 \rightarrow$ Adds a pole to $s = -1$

Example 2: $\gamma = 0.22 \rightarrow$ Adds a pole to $s = -4.5$.

Example 3: $\gamma = 0.10 \rightarrow$ Adds a pole to $s = -10$.

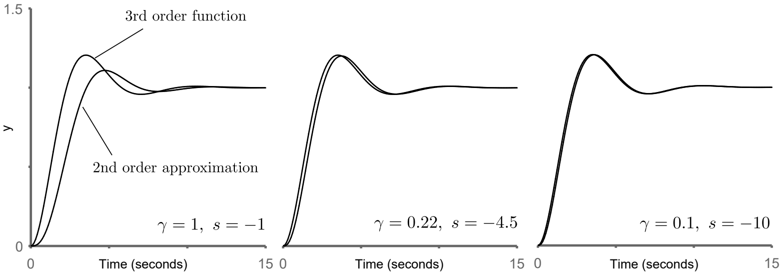
Second order systems with an additional pole

Original 3rd order function:

$$T(s) = \frac{1}{(s^2 + 2\zeta\omega_n s + 1)(\gamma s + 1)}.$$

2nd order approximation:

$$T_a(s) = \frac{1}{s^2 + 2\zeta\omega_n s + 1}.$$



Additional zeros

Consider the transfer function with an additional zero $s = -z$:

$$\frac{Y(s)}{R(s)} = \frac{\frac{\omega_n^2}{z}(s+z)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2}$$

If $z >> \zeta\omega_n$, the zero will have minimal effect on the step response.

The unit step response of the above equation is:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{\frac{\omega_n^2}{z}s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{3}$$

If $x(t)$ is the inverse of the first term, then the time response is

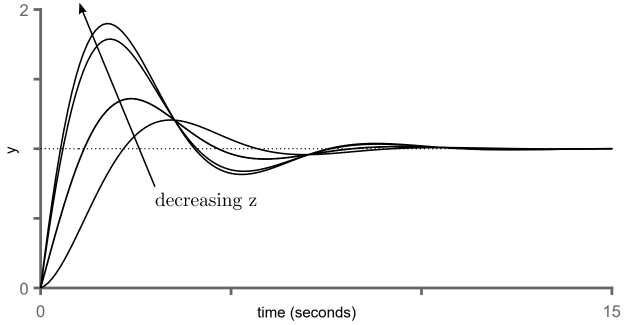
$$y(t) = x(t) + \frac{1}{z}\left(\frac{d}{dt}x(t)\right) \tag{4}$$

Conclusion: The additional zero speeds us transients, making rises and falls sharper.

Additional zeros

$$\frac{Y(s)}{R(s)} = \frac{\frac{\omega_n^2}{z}(s+z)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5}$$

Consider: $\omega_n = 1,\ \zeta = 0.45,\ z = 0.7,\ 1,\ 10$



Simplification to a lower order

A more precise approach: Match the frequency response.

Consider the high order system:

$$G_H(s) = K\frac{a_m s^m + a_{m-1}s^{m-1} + \ldots + a_1 s + 1}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + 1} \tag{6}$$

with $m \geq n$, which is to be mapped to a lower order system

$$G_L(s) = K\frac{c_p s^p + c_{p-1}s^{p-1} + \ldots + c_1 s + 1}{d_g s^g + d_{g-1}s^{g-1} + \ldots + d_1 s + 1} \tag{7}$$

such that $p \leq g \leq n$.

The c and d coefficients of the approximate solution G_L are obtained via

$$M^k = \frac{d^k}{ds^k}M(s) \tag{8}$$

$$\Delta^k = \frac{d^k}{ds^k}\Delta(s) \tag{9}$$

Simplification

Let us define

$$M_{2q} = \sum_{k=0}^{2q} \frac{(-1)^{k+q} M^k(0) M^{2q-k}(0)}{k!(2q-k)!} \tag{10}$$

$$\Delta_{2q} = \sum_{k=0}^{2q} \frac{(-1)^{k+q} \Delta^k(0) \Delta^{2q-k}(0)}{k!(2q-k)!} \tag{11}$$

So that the c and d coefficient are obtained by equating

$$M_{2q} = \Delta_{2q} \tag{12}$$

for $q = 1, 2...$ and up to the number required to solve for the unknowns.

Location of poles

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

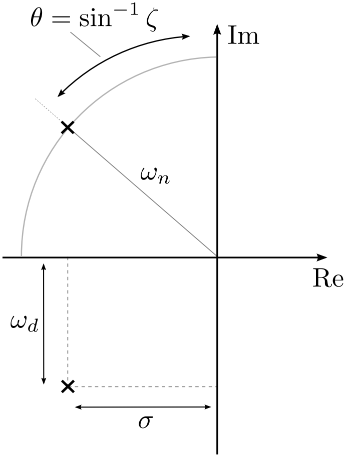
The poles are

$$s = \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$s = -\sigma \pm j\omega_d$$

where $\sigma = \zeta\omega_n$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$

→ Poles are located at a radius ω_n

→ The angle to the imaginary axis is $\theta = \sin^{-1}\zeta$



Exercise 40

A closed-loop control system has a transfer function $T(s)$ as follows

$$T(s) = \frac{Y(s)}{R(s)} = \frac{2500}{(s+50)(s^2+10s+50)}.$$

Plot the time response to an unit step input when:

→ **(a)** The actual $T(s)$ is used (use Matlab)

→ **(b)** Using the dominant complex poles

→ **(c)** Compare the results

Exercise 40 - continued

(a) The actual function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{2500}{(s+50)(s^2+10s+50)}.$$

(b) The approximate transfer function is

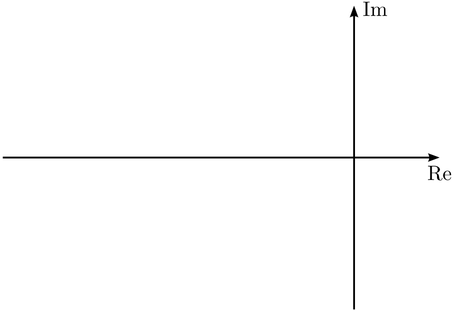
Exercise 41

A closed-loop control system transfer function as two dominant complex conjugate poles. Sketch the region in the left-hand s-plane where the complex poles should be located to meet the given specifications:

- → **(a)** $0.6 \leq \zeta \leq 0.8, \quad \omega_n \leq 10$
- → **(b)** $0.5 \leq \zeta \leq 0.707, \quad \omega_n \geq 10$
- → **(c)** $\zeta \geq 0.5, \quad 5 \leq \omega_n \leq 10$
- → **(d)** $\zeta \leq 0.707, \quad 5 \leq \omega_n \leq 10$
- → **(e)** $\zeta \geq 0.6, \quad \omega_n \leq 6$

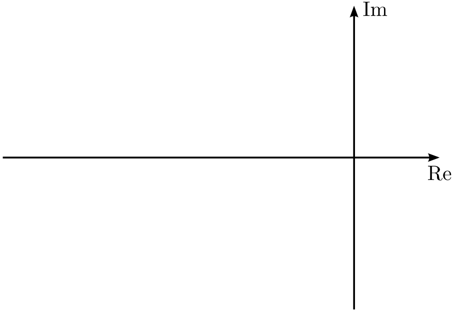
Exercise 41 - continued

→ **(a)** $0.6 \leq \zeta \leq 0.8, \qquad \omega_n \leq 10$



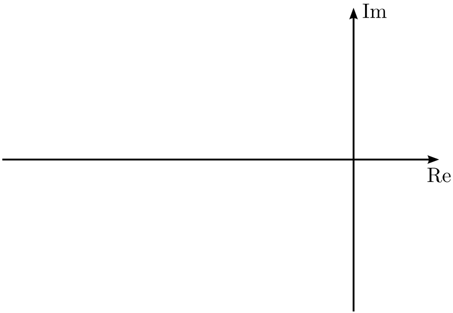
Exercise 41 - continued

→ **(b)** $0.5 \leq \zeta \leq 0.707, \quad \omega_n \geq 10$



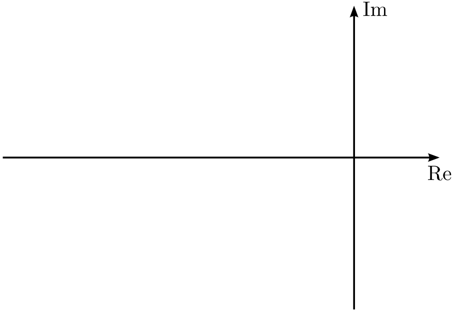
Exercise 41 - continued

→ **(c)** $\zeta \geq 0.5$, $5 \leq \omega_n \leq 10$



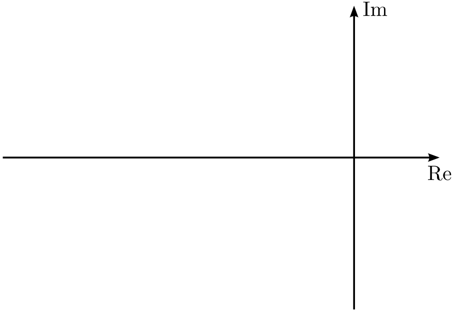
Exercise 41 - continued

→ **(d)** $\zeta \leq 0.707$, $5 \leq \omega_n \leq 10$



Exercise 41 - continued

→ **(e)** $\zeta \geq 0.6$, $\omega_n \leq 6$



Exercise 42

A closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)}.$$

→ **(a)** Determine the steady state error for a unit step input.

→ **(b)** Assume that the complex poles dominate and determine the percent overshoot an setting time.

→ **(c)** Plot the actual system response and compare it with **(b)**

Exercise 42 - continued

(a) Steady-state error for $r(t) = 1$.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)}.$$

Exercise 42 - continued

(b) Overshoot and settling time considering the dominant poles.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)}.$$

Exercise 42 - continued

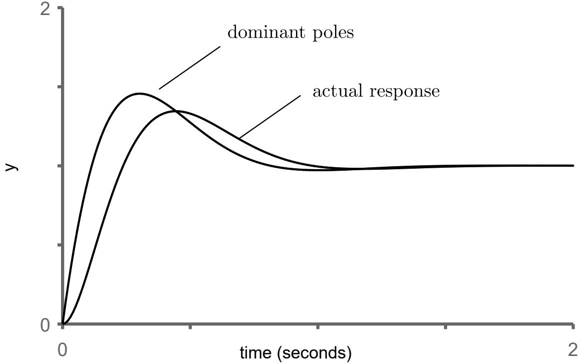
(c) Overshoot and settling time considering the dominant poles.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)}.$$

Exercise 42 - continued

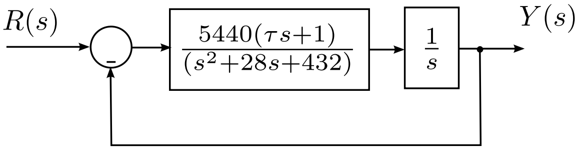
```
T = tf([108 324],[1 17 108 324]);
step(T); stepinfo(T)
```

```
H = tf([108/9 324/9],[1 8 36]);
step(H); stepinfo(H)
```



Exercise 43

Consider the following closed loop system



Where τ can take the values $\tau = 0,\ 0.05,\ 0.1$ or 0.5. For $r(t) = 1$:

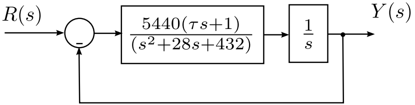
→ **(a)** Record the percent overshoot, rise time, and settling time as τ varies.

→ **(b)** Describe the effects of varying τ.

→ **(c)** Compare the location of the zero with that of the closed-loop poles.

Exercise 43 - continued

The closed loop transfer function



Exercise 43 - continued

$$T(s) = \frac{5440(\tau s + 1)}{s^3 + 28s^2 + (432 + 5440\tau)s + 5440}$$

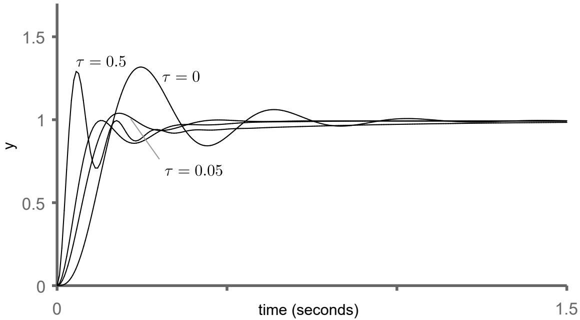
Matlab commands:

```
H = tf([5440*t 5400],[1 28 432+5440*t 5440]);
infostep(H)
damp(H)
```

τ	T_r	T_s	P.O.	zero	pole
0					
0.05					
0.1					
0.5					

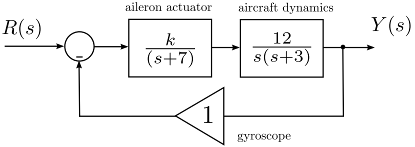
Exercise 43 - continued

```
t = 0;
H1 = tf([5440*t 5400],[1 28 432+5440*t 5440]);
step(H1);
```



Exercise 44

The roll control of an aircraft is shown. The goal is to select a suitable K so that the response to a step command $r(t) = A$ will provide a fast response with an overshoot of less than 20%.



Steps for designing the controller:

→ **(a)** Determine the closed-loop transfer function

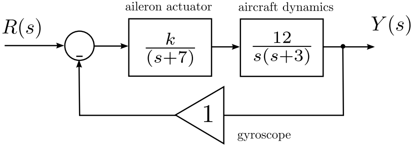
→ **(b)** Determine the poles for $K = 0.7$, 3, and 6;

→ **(c)** Using the concept of dominant poles find the expected overshoot

→ **(d)** Plot the actual response with Matlab and compare it with (c)

Exercise 44 - continued

(a) The closed-loop transfer function



Exercise 44 - continued

(b) Finding the poles

$$T(s) = \frac{12k}{s(s+3)(s+7)+12k} = \frac{12k}{s^3+10s^2+21s+12k} \tag{13}$$

Exercise 44 - continued

(c) Overshoot considering the dominant poles (k $=$0.7, 3, and 6).

$$T(s) = \frac{12k}{s(s+3)(s+7)+12k} = \frac{12k}{s^3+10s^2+21s+12k} \tag{14}$$

Exercise 44 - continued

(d) Step-unit response using Matlab

Exercise 44 - continued

(c) Overshoot considering the dominant poles (k =0.7, 3, and 6).

$$T(s) = \frac{12k}{s(s+3)(s+7)+12k} = \frac{12k}{s^3+10s^2+21s+12k} \tag{15}$$

Skills check 25 - From 2018 midterm examination

True or false (1 mark)? The complex poles of P(s) will dominate its response to a unit step input. Justify your answer (4 marks).

$$P(s) = \frac{150}{(s^2 + 3s + 10)(s + 15)} \tag{16}$$

For answer see last slide

Skills check 26 - From 2018 midterm examination

Based on the given unit step response $h(t)$ of the function $H(s)$ shown below, what temporal signal best describes the unit response of $G(s)$?

$$H(s) = \frac{10}{s^2 + 2s + 10}, \quad G(s) = \frac{10(s+1)}{s^2 + 2s + 10}$$

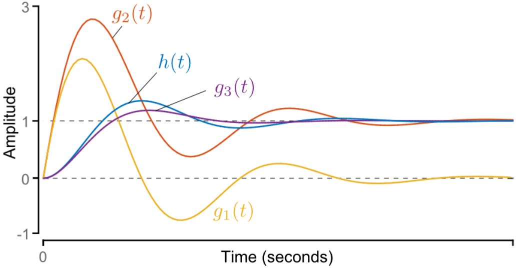
(a) $g_1(t)$

(b) $g_2(t)$

(c) $g_3(t)$

(d) $g(t) \approx h(t)$

(e) None of the above



Next class...

- Stability
- Solution to skills check: 25 - True, 26 - (b) (why not (a)?).