MECE 3350U
Control Systems

Lecture 10

Routh-Hurwitz Stability Criterion

Videos in this lecture

Lecture: https://youtu.be/EJMznJZZEbw

Exercise 45: https://youtu.be/ombQ12xaAKo

Exercise 46: https://youtu.be/8ZfVH51bRz4

Exercise 47: https://youtu.be/9PeGQF-jwDI

Exercise 48: https://youtu.be/BVSfi1i0_nE

Exercise 49: https://youtu.be/0IYWnkNEkWA

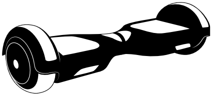
Outline of Lecture 10

By the end of today's lecture you should be able to

- Understand the principle of stability
- Define the conditions required for stability
- Apply to the Routh-Hurwitz stability criterion

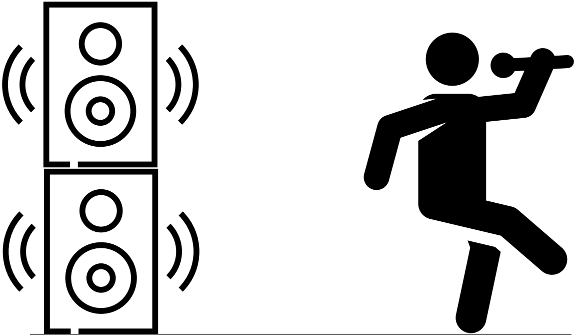
Applications

Are these control systems stable without feedback control?

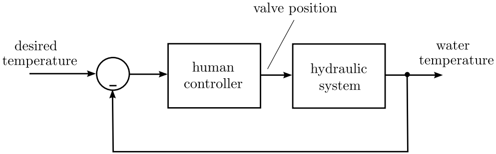


Applications

Example of destabilizing positive feedback.

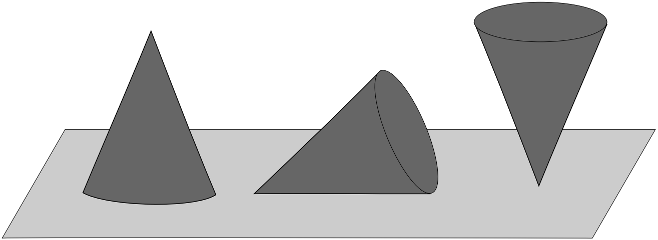


High control loop gains can make a system unstable.



The concept of stability

A stable system is a dynamic system with a bounded response to a bounded input.



A system is stable is all closed-loop transfer function poles lie in the left-half $s-$plane.

Requirements for stability

Consider the generic transfer function:

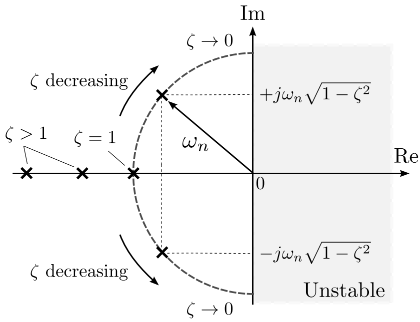
$$T(s) = \frac{p(s)}{q(s)} = \frac{k \prod_{i=1}^{M}(s + z_i)}{s^N \prod_{k=1}^{Q}(s + \sigma_k) \prod_{m=1}^{R}\left[s^2 + 2\alpha_m s + (\alpha_m^2 + \omega_m^2)\right]} \qquad (1)$$

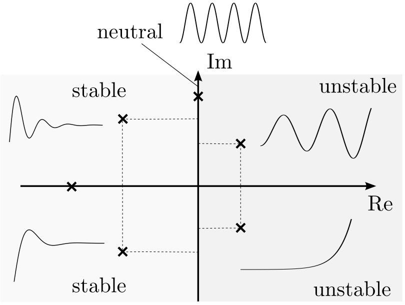
The output response for an impulse function input and $N = 0$ is

$$y(t) = \sum_{k=1}^{Q} A_k e^{-\sigma_k t} + \sum_{m=1}^{R} B_m \left(\frac{1}{\omega_m}\right) e^{-\alpha_m t} \sin(\omega_m t + \theta_m) \qquad (2)$$

A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have **negative real parts**

Stability and the location of poles





Stable system: Poles are in the left-half plane

Neutral system or marginally stable: Poles are purely imaginary ($j\omega$)

Unstable system: At least one of the poles is in the right-half plane

Marginally stable

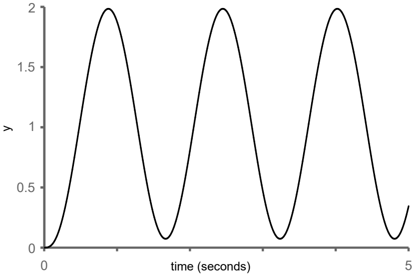
Some poles of the closed-loop transfer functions are purely imaginary.

Example: The transfer function

$$T(s) = \frac{1}{(s+10)(s^2+16)} \tag{3}$$

has the poles $s_1 = -10$, $s_2 = 4j$, $s_3 = -4j$.

For $r(t) = 1$:



The Routh-Hurwitz criterion

This criterion is a necessary and sufficient condition for stability

Order the coefficient of the characteristic equation

$$\Delta(s) = q(s) = a_n s^n + a_{n-1} s^{s-1} + \ldots + a_1 s + a_0 = 0 \tag{4}$$

Into an array as follows:

$$\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \\
s_0 & h_{n-1} & & &
\end{array} \tag{5}$$

The number of roots with positive real parts is equal to the number of changes in sign of the first column.

The Routh-Hurwitz criterion

Step 1: Place the highest order of $q(s)$ on the top-left column from n to 0.

$$\begin{array}{|c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \dots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \tag{6}$$

Step 2: From the second column, the first two rows are the coefficients of

$$\Delta(s) = q(s) = a_n s^n + a_{n-1} s^{s-1} + \ldots + a_1 s + a_0 = 0$$

$$\begin{array}{c|cccc|} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ \hline s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \dots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \tag{7}$$

The Routh-Hurwitz criterion

Step 3: Fill out the reminder rows

$$\begin{array}{lllll} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ s^{n-2} & \boxed{b_{n-1}} & \boxed{b_{n-3}} & \boxed{b_{n-5}} & \boxed{\dots} \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \tag{8}$$

For the b_n coefficients:

$$b_{n-1} = \frac{-1}{a_{n-1}} \left\| \begin{array}{cc} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{array} \right\| \tag{9}$$

$$b_{n-3} = \frac{-1}{a_{n-1}} \left\| \begin{array}{cc} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{array} \right\| \tag{10}$$

The Routh-Hurwitz criterion

Step 3: Fill out the remainder rows

$$\begin{array}{lllll} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \dots \\ s^{n-3} & \boxed{c_{n-1} \quad c_{n-3} \quad c_{n-5} \quad \dots} & & & \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \tag{11}$$

For the c_n coefficients:

$$c_{n-1} = \frac{-1}{b_{n-1}} \left\| \begin{array}{ll} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{array} \right\| \tag{12}$$

$$c_{n-3} = \frac{-1}{b_{n-1}} \left\| \begin{array}{ll} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{array} \right\| \tag{13}$$

And so on...

Example of Routh-Hurwitz matrix

$$q(s) = s^5 + 2s^4 + 1s^3 + 4s^2 + 11s + 10$$

The Routh-Hurwitz matrix is

s^5	1	1	11
s^4	2	4	10
s^3			
s^2			
s^1			
s^0			

$$b_{n-1} = \frac{-1}{a_{n-1}} \begin{Vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{Vmatrix}$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{Vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{Vmatrix}$$

Example of Routh-Hurwitz matrix

$$q(s) = 3s^5 + s^4 + 2s^3 + 1s^2 + 1$$

The Routh-Hurwitz matrix is

s^5	3	2	0
s^4	1	1	1
s^3			
s^2			
s^1			
s^0			

$$b_{n-1} = \frac{-1}{a_{n-1}} \begin{Vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{Vmatrix}$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{Vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & a_{n-3} \end{Vmatrix}$$

The Routh-Hurwitz Criterion

Stability requires all roots of $q(s)$ to have positive real parts.

$\rightarrow$ **Count the sign *changes* in the first column**

$\rightarrow$ **That is the number of roots in the right half plane**

Case 1: All elements in the first column are nonzero

$$q(s) = a_2 s + a_1 s + a_0$$

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s_1 & a_1 & 0 \\ s_0 & a_0 & 0 \end{array}$$

Thus the system is stable if:

$\rightarrow$ $a_2 > 0$, **and** $a_1 > 0$, **and** $a_0 > 0$.

or

$\rightarrow$ $a_2 < 0$, **and** $a_1 < 0$, **and** $a_0 < 0$.

The Routh-Hurwitz Criterion

Case 2: There is a zero in the first column. Other elements in the row containing the zero are nonzero.

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Replace the zero in the first column with $\epsilon \to 0^+$, i.e. $0 < \epsilon << 1$

s^5	1	2	11
s^4	2	4	10
s^3	ϵ	6	0
s^2	c_1	10	0
s_1	d_1	0	0
s_0	10	0	0

where:

$$c_1 = \frac{4\epsilon - 12}{\epsilon}, \qquad d_1 = \frac{6c_1 - 10\epsilon}{c_1}$$

Now, make $\epsilon \to 0^+$ and evaluate the first column elements signs

The Routh-Hurwitz Criterion

Case 3: There is a zero in the first column and the other elements of the row containing the zero are also zero.

Case 4: Repeated roots of the characteristic equation on the imaginary axis.

What does it mean?

⇒ The characteristic equations has purely imaginary roots

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & k \\ s_1 & \frac{8-k}{2} & \\ s_0 & k & 0 \end{array}$$

Exercise 45

Find the Routh-Hurwitz matrix for the following closed-loop transfer functions and assess the stability of each system.

$$T(s) = \frac{1}{s^2 + 2s + 1}$$

$$R(s) = \frac{1 + s}{3s^3 + s^2 + 2s + 3}$$

$$P(s) = \frac{1}{s^4 + 2s^3 - 100s - 500}$$

$$H(s) = \frac{s^2 + 1}{s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63}$$

Exercise 45 - continued

$$T(s) = \frac{1}{s^2 + 2s + 1}$$

Exercise 45 - continued

$$R(s) = \frac{1+s}{3s^3 + s^2 + 2s + 3}$$

Exercise 45 - continued

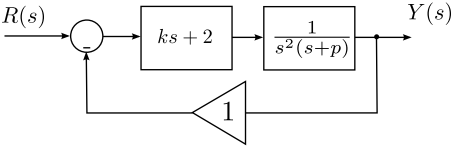
$$P(s) = \frac{1}{s^4 + 2s^3 - 100s - 500}$$

Exercise 45 - continued

$$H(s) = \frac{s^2 + 1}{s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63}$$

Exercise 46

A closed-loop feedback system is shown in the figure.



For what range of values of the parameters k and p is the system stable ?

Procedure:

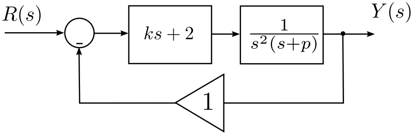
$\rightarrow$ Find the closed-loop transfer function

$\rightarrow$ Write the Routh-Hurwitz matrix

$\rightarrow$ Determine the values of p and k that meet the stability condition

Exercise 46 - continued

Step 1 - Find the closed-loop transfer function



Exercise 46 - continued

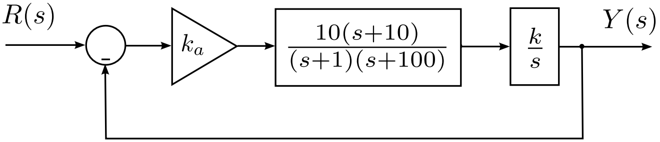
Step 2 - Find the Routh-Hurwitz matrix

$$T(s) = \frac{ks+2}{s^2(s+p)+ks+2} = \frac{ks+2}{s^3+ps^2+ks+2}$$

s^3	
s^2	
s^1	
s^0	

Exercise 47

The linear model of a phase detector can be represented by the diagram shown. It is designed to maintain a zero phase between the input carrier signal and local voltage controller oscillator. We want to minimize th steady-state error for a ramp input.

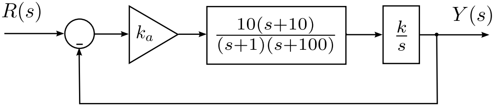


(a) Determine the maximum gain $k_a k$ in order to maintain a stable system.

(b) Find kk_a for a steady-state error of 1° for a ramp signal of 100 rad/s.

Exercise 47 - continued

Step 1 - Find the closed-loop characteristic equation.



Exercise 47 - continued

Step 2: Find the Routh matrix

$$q(s) = s^3 + 101s^2 + (100 + 10kk_a)s + 100kk_a = 0$$

s^3	
s^2	
s^1	
s^0	

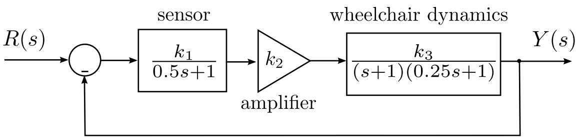
Exercise 47 - continued

(c) Find kk_a that yield a tracking error of 1° for $r(t) = 100t$

$$e(s) = \lim_{s \to 0} s(1 - T(s)) \frac{100}{s^2}$$

Exercise 48

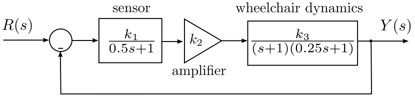
A wheelchair velocity control system is shown in the diagram.



Determine the maximum gain $k_1 k_2 k_3$ for a stable system.

Exercise 48 - continued

The closed loop transfer function



Exercise 48 - continued

$$q(s) = s^3 + 7s^2 + 14s + 8(1 + k) = 0$$

The Routh matrix is

$$\begin{array}{c|} s^3 \\ s^2 \\ s^1 \\ s^0 \end{array}$$

Stability requires that:

Exercise 49

A teleoperated control system incorporates both a person and a remote machine. In the case of remote operation of a robot, force feedback is useful. The characteristic equation of such a system is

$$s^4 + 20s^3 + k_1 s^2 + 4s + k_2 = 0$$

where k_i is a feedback force amplification factor. Determine and plot the region of stability for this system for k_1 and k_2.

Exercise 49 - continued

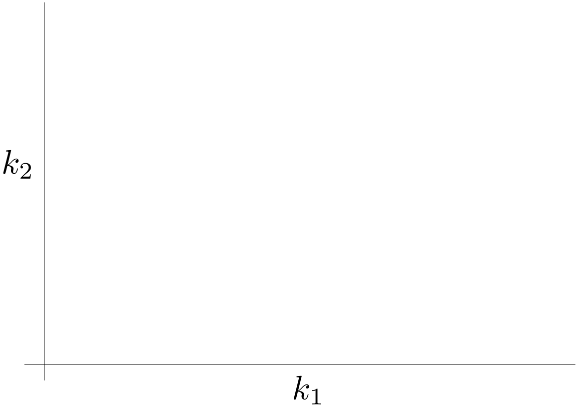
$$s^4 + 20s^3 + k_1 s^2 + 4s + k_2 = 0$$

The Routh array is

s^4	
s^3	
s^2	
s^1	
s^0	

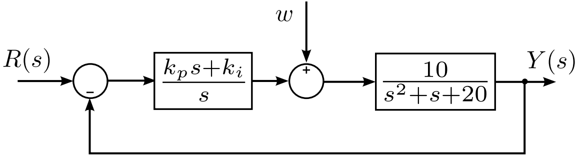
Exercise 49 - continued

For stability: $k_2 > 0$, $k_1 > 0.2$, and $k_2 < 0.2k_1 - 0.04$



Skills check 27 - From a midterm examination

A laparoscopic robot arm operating under feedback control has the following structure.



A controller having two gains k_p and k_i is used to regulate the arm's position $Y(s)$ as a function of the desired position $R(s)$. Neglect the disturbance and assume that $w = 0$.

(a) Find the characteristic equation of the closed-loop system.

(b) Use Routh's criterion to find the range of k_p and k_i for which the system is stable (continued on next slide).

Skills check 27 - continued

(c) In the graph below, the axes are k_p and k_i. Identify the region where the combination of these gains results in a stable system.

Answer: Stable for $k_p > k_i - 2$ with $k_i > 0$

Skills check 28 - From a past examination

If $H(s)$ is a **closed-loop** transfer function, what is the maximum gain k that results in a stable closed-loop system ? (*3 marks*)

$$H(s) = \frac{k}{s^3 + 2s^2 + s + k}$$

Answer: Stable for $k < 2$

Skills check 29 - From a final examination

If $H(s)$ is a **open-loop** transfer function, what is the maximum gain k that results in a stable **closed-loop** system ? (*3 marks*)

$$H(s) = \frac{k}{s^3 + 2s^2 + s + k}$$

Answer: Stable for $k < 1$

Skills check 30 - From a deferred final examination

If $H(s)$ is a **open-loop** transfer function, what is the maximum gain k that results in a stable **closed-loop** system ? (*3 marks*)

$$H(s) = \frac{k}{s^5 + s^4 + 2s^2 + s + k}$$

Answer: Unstable $\forall\ k$

Next class...

- Root-locus method