MECE 3350U
Control Systems

Lecture 15

Midterm Examination Review and Practice Exercises

Videos in this lecture

Lecture: `https://youtu.be/yo15W7-hJ5k`

Exercises 73 to 92: Only the final answer is provided.

Use these extra exercises to practice for the upcoming midterm.

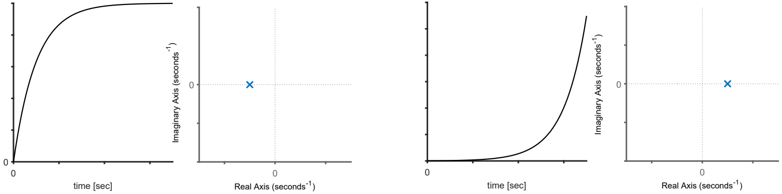
First order transfer functions

First order functions are written in the from

$$T(s) = \frac{k}{s+\sigma}$$

where $\tau = \frac{1}{\sigma}$ is called the time constant. The response to an unit step response is

$$y(t) = k\left(1 - e^{-\sigma t}\right)$$



If $\sigma > 0$, the pole is on the left-half s-plane.

If $\sigma < 0$, the pole is on the right-half s-plane.

Second order response

A second order system is typically represented as

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\Rightarrow \zeta$ is the damping ratio

$\Rightarrow \omega_n$ is the undamped natural frequency

The poles of the transfer function are:

$$s_1 = \omega_n \left(-\zeta + \sqrt{\zeta^2 - 1}\right)$$
$$s_2 = \omega_n \left(-\zeta - \sqrt{\zeta^2 - 1}\right)$$

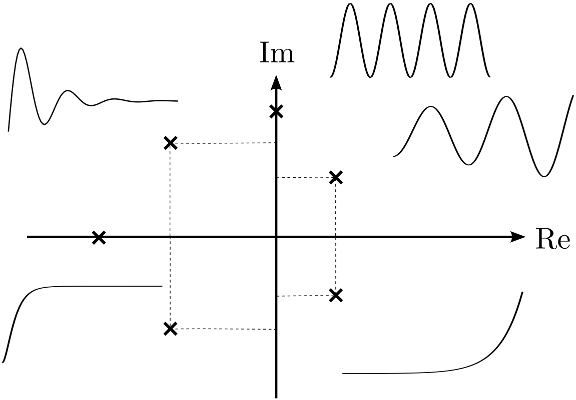
$\zeta > 1$ Overdamped system

$0 < \zeta < 1$ Underdamped system

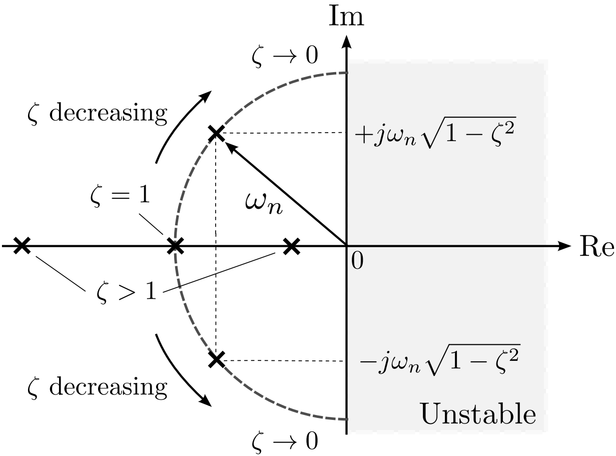
$\zeta = 1$ Critically damped system

$\zeta = 0$ Undamped system, $\zeta < 0$ Unstable

Summary

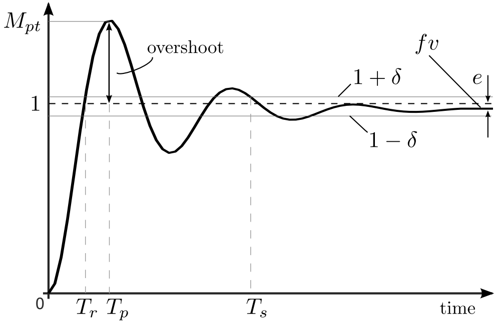


Summary



Im
$\zeta \to 0$
ζ decreasing
$+j\omega_n\sqrt{1-\zeta^2}$
$\zeta = 1$
ω_n
Re
0
$\zeta > 1$
$-j\omega_n\sqrt{1-\zeta^2}$
ζ decreasing
Unstable
$\zeta \to 0$

Performance of feedback control systems



- → Steady estate error
- → Rise time T_r, peak time T_p, and peak value M_{pt}
- → Settling time T_s: $y(t)$ within 2% of its final value
- → Percent overshoot $P.O.$

Performance of feedback control systems

Peak time

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}}$$

Magnitude at the peak time

$$M_{pt} = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

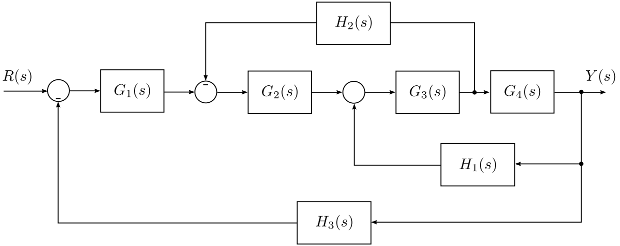
Percentage overshoot

$$P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

Settling time

$$T_s = \frac{4}{\zeta\omega_n} = 4\tau$$

Block diagrams



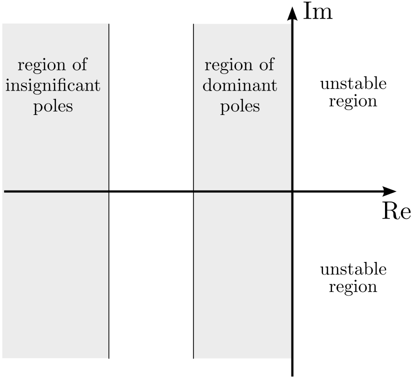
The tree fundamental operations are

→ Obtain a block diagram from a transfer function

→ Obtain a transfer function from a block diagram

→ Simplify a block diagram

Dominant poles



If the magnitude of the real part of a pole is at least 5 to 10 times that of a dominant pole, then the pole may be regarded as insignificant.

The Routh-Hurwitz criterion

This criterion is a necessary and sufficient condition for stability

Order the coefficient of the characteristic equation

$$\Delta(s) = q(s) = a_n s^n + a_{n-1} s^{s-1} + \ldots + a_1 s + a_0 = 0 \tag{1}$$

Into an array as follows:

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \tag{2}$$

The number of roots with positive real pats is equation to the number of changes in sign of the first column.

The Routh-Hurwitz criterion

Step 1: The highest order of $q(s)$ goes on the top-left column from n to 0.

Step 2: From the second column, the first two rows are the coefficients of the characteristic equation

$$\begin{array}{l|llll|} \cline{2-5} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ \cline{2-5} s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \dots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array} \quad (3)$$

Step 3: Fill out the reminder rows

$$\begin{array}{l|llll|} s^n & a_n & a_{n-2} & a_{n-4} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ \cline{2-5} s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \dots \\ \cline{2-5} s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \\ s_0 & h_{n-1} & & & \end{array}$$

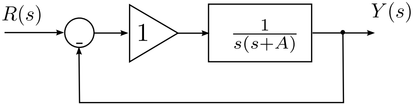
$$b_{n-1} = \frac{-1}{a_{n-1}} \begin{Vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{Vmatrix}$$

The root locus method

How does the location of the poles of a transfer function with characteristic equation

$$1 + kL(s)$$

change, as k goes from 0 to infinity?



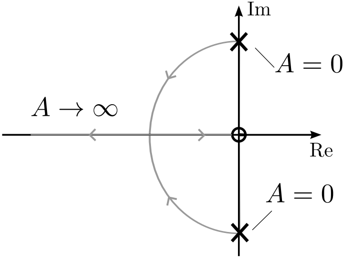
The characteristic equations is

$$1 + \frac{1}{s(s+A)} = 0$$

$$s^2 + As + 1 = 0; \rightarrow (s^2+1) + As = 0$$

$$\frac{(s^2+1)}{(s^2+1)} + A\frac{s}{(s^2+1)} = 0$$

$$1 + A\frac{s}{s^2+1} = 0$$



Steps for drawing the root locus

Step 1 Prepare the characteristic equation in the form of

$$1 + kH(s) = 0 \tag{4}$$

Step 2 Locate the poles and zeros of $H(s)$ in the plane

Step 3 Locate the segments of the of the real axis that are root loci. Root loci are to the left of an odd number of poles and zeros.

Step 4 Calculate the angle θ and centre α of asymptotes of loci that tend to infinity

$$\theta = \frac{180^\circ + 360^\circ(q-1)}{n-m} \qquad \alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

Step 5 Determine the points at which the loci cross the imaginary axis. Use Routh-Hurwitz criterion.

Step 6 Determine the breakaway point on the real axis.

Steps for drawing the root locus

Step 7 Determine the angle of locus departure from complex poles and the angle of locus at arrival at complex zeros using the phase criterion.

$$q\phi = \sum \psi - \sum \phi - 180^\circ - \ell 360^\circ$$

$$q\psi = \sum \phi - \sum \psi + 180^\circ + \ell 360^\circ$$

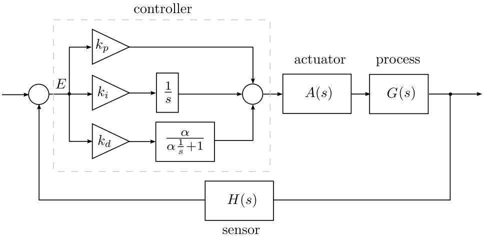
Step 8

Complete the root locus

Step 9

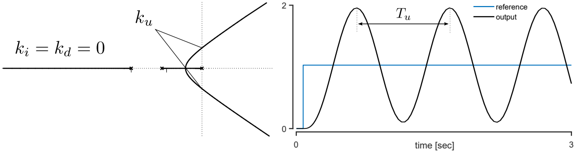
You may check you results using the Matlab function "rlocus(H);".

PID controller



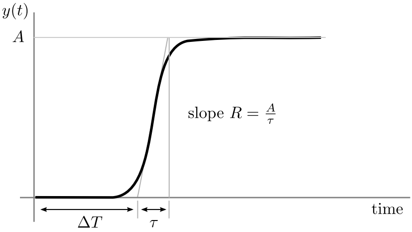
PID gain	Overshoot	Settling time	Steady-state error
Increasing k_p	Increases	Minimal impact	Decreases
Increasing k_i	Increases	Increases	Zero error
Increasing k_d	Decreases	Decreases	No impact

Ziegler-Nichols PID tuning - Method 1



Controller type	k_p	k_i	k_d
Proportional $C(s) = k_p$	$0.5k_u$	0	0
Proportional-integral $C(s) = k_p + k_i s^{-1}$	$0.45k_u$	$\frac{0.54k_u}{T_u}$	0
PID $C(s) = k_p + k_i s^{-1} + k_d s$	$0.6k_u$	$\frac{1.2k_u}{T_u}$	$\frac{0.6K_u T_u}{8}$

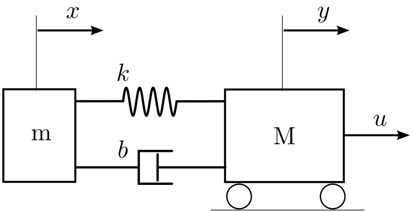
Ziegler-Nichols PID tuning - Method 2



Controller type	k_p	k_i	k_d
Proportional $C(s) = k_p$	$\frac{1}{R\Delta T}$	0	0
Proportional-integral $C(s) = k_p + k_i s^{-1}$	$\frac{0.9}{R\Delta T}$	$\frac{0.27}{R\Delta T^2}$	0
PID $C(s) = k_p + k_i s^{-1} + k_d s$	$\frac{1.2}{R\Delta T}$	$\frac{0.6}{R\Delta T^2}$	$\frac{0.6}{R}$

Exercise 73

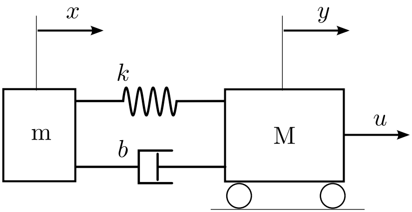
In the system shown, a force u is applied to the mass M and another m is connected to it. The coupling between the objects is often modelled by a spring constant k with a damping coefficient b. Write the equations of motion in the Laplace domain. [1]



[1] $m\ddot{x} = -k(x-y) - b(\dot{x}-\dot{y})$
$M\ddot{y} = u + k(x-y) + b(\dot{x}-\dot{y})$

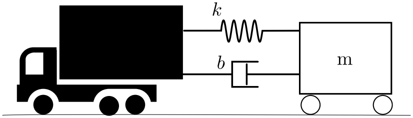
Exercise 74

Based on the equations obtained in Exercise 68, draw a block diagram for the system of two masses.



Exercise 75

Find the transfer function between the position of the truck and the position of the cart. [2]



[2] $T(s) = (bs + k)/(ms^2 + bs + k)$

Exercise 76

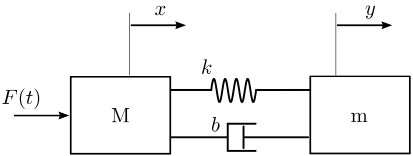
Without computing the inverse transformation, sketch the temporal response of the following transfer functions to a step input. Specify the steady state value. Verify your plots using Matlab.[3]

$$T(s) = \frac{1}{s^2 + s + a}$$
$$D(s) = \frac{1}{s^2 + 5s + 1}$$
$$R(s) = \frac{1}{s^2 + 2}$$
$$H(s) = \frac{50}{s^2 + 15s + 50}$$

[3] Solutions can be found using Matlab

Exercise 77

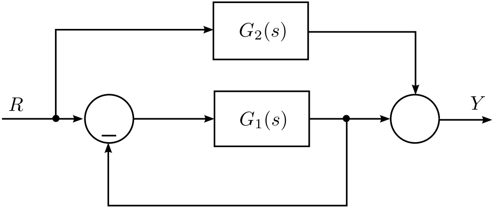
A robot includes significant flexibility in the arm members with a heavy load in the gripper. A two-mass model of the robot is shown in the figure. Find the transfer function $Y(s)/F(s)$.[4]



[4] $T(s) = \dfrac{\frac{1}{mM}(bs+k)}{s^2\left[s^2+(1+\frac{m}{M})(\frac{b}{m}s+\frac{k}{m})\right]}$

Exercise 78

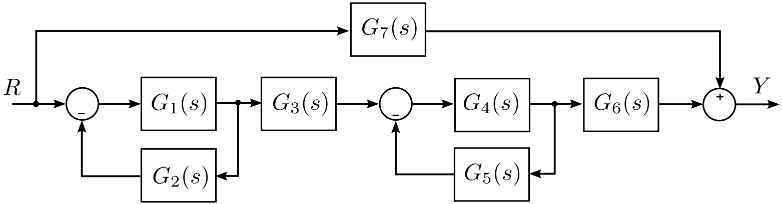
Find the transfer function $Y(s)/R(s)$ for the block diagram shown.[5]



[5] $T(s) = \frac{G_1}{1+G_1} + G_2$

Exercise 79

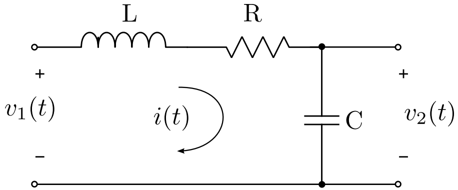
Find the transfer function $Y(s)/R(s)$ for the block diagram shown.[6]



[6] $T(s) = G_7 + \frac{G_1 G_3 G_4 G_6}{(1+G_1 G_2)(1+G_4 G_5)}$

Exercise 80

Consider the LRC circuit shown.



Find the following:

(a) The time domain equation relating $i(t)$ and $v_1(t)$

(b) The time domain equation relating $i(t)$ and $v_2(t)$

(c) The transfer function $V_2(s)/V_1(s)$

(d) The circuit damping ration and the natural frequency

(e) The value of R that results in $v_2(t)$ having an overshoot no more than 25% for an unit step of $v_1(t)$. Take $L = 10$ mH, $C = 4\mu$F.

Exercise 80 - continued

Solution

(a) $v_1(t) = L\frac{di(t)}{dt} + Ri + \frac{1}{C} i(t)dt$

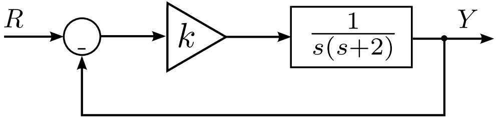
(b) $v_2(t) = \frac{1}{C} i(t)dt$

(c) $\frac{V_2(s)}{V_1(s)} = \frac{1}{s^2LC+sRC+1}$

(e) For 25% overshoot, $\zeta = 0.4$ and thus $R = 40\Omega$

Exercise 81

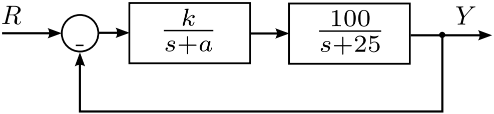
For the unit feedback closed-loop system shown, specify the proportional controller gain k so that the output $y(t)$ has an overshoot of no more than 10% in response to a unit step.[7]



[7] $\zeta \geq 0.591$, thus $0 < k \leq 2.86$

Exercise 82

For the unit feedback closed-loop system shown, specify the proportional controller gain k and the location of the pole a so that the output $y(t)$ has an overshoot of no more than 25%, and a settling time of no more than 0.1 sec in response to a unit step.[8]



Verify your results using Matlab.

[8] $\zeta \geq 0.4037$, $\omega_n \approx 99$, thus $a = 54.99$, and $k \approx 84.75$.

Exercise 83

Two closed-loop transfer functions are given below.

$$\frac{Y(s)}{R(s)} = \frac{2}{s^2 + 2s + 2}$$

$$\frac{Y(s)}{R(s)} = \frac{2s + 6}{2(s^2 + 2s + 2)}$$

In each case, provide estimates of the settling time and percent overshoot to a unit input in $r(t)$.[9]

[9] $t_s = 4$ sec, $P.O. = 4.32\%$ sec, $t_s = 1.3$ sec, 5%

Exercise 84

Using Routh's stability criterion, determine how many roots with positive real parts the following equations have.[10]

(a) $s^4 + 8s^3 + 32s^2 + 80s + 100 = 0$

(b) $s^4 + 2s^3 + 7s^2 - 2s + 8 = 0$

(c) $s^3 + s^2 + 20s + 78 = 0$

(d) $s^4 + 6s^2 + 25 = 0$

[10] Use the provided Matlab code to check your answers.

Exercise 85

The transfer function of a typical hard drive system is given by

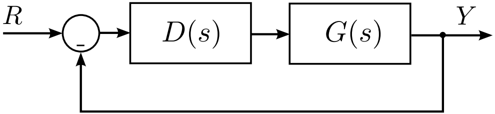
$$G(s) = \frac{k(s+4)}{s(s+0.5)(s+1)(s^2+0.4s+4)}$$

Using Routh's stability criterion, determine the range of k for which this system is stable when the characteristic equation is $1 + G(s) = 0$.[11]

[11] $0 < k < 0.78$

Exercise 86

Consider the following closed-loop system



where

$$G(s) = \frac{1}{s}, \qquad D(s) = \frac{k}{s+p}$$

Find k, and p so that the system has a 10% overshoot to a step input and a settling time of 1.5 sec.[12]

[12] $\zeta = 0.7,\ k = 19.75,\ p = 5.3$

Exercise 87

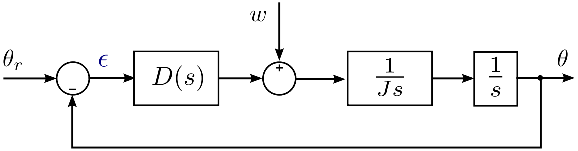
Consider the satellite altitude controller shown where the parameters are

$J = 10$ space craft inertia (N·m·sec^2/rad),

θ_r reference satellite altitude (rad)

θ actual satellite altitude (rad)

w disturbance torque (N·m)



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Exercise 87 - continued

(a) Use propositional controller ($D(s) = k$) and evaluate the stability of the system.

Determine the steady-state value of θ for the following scenarios

(b) Using PD control and a unit step reference input.

(c) Using PD control and a unit disturbance step input.

(d) Using PI control control and a unit step reference input.

(e) Using PI control and a unit disturbance step input.

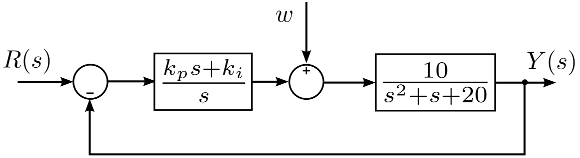
(f) Using PID control and a unit step reference input.

(g) Using PID control and a unit disturbance step input.

(a) The system is unstable, (b) 1 rad, (c) $1/(k_p)$, (d) stable, (e) unstable, (f) 1 rad, (g) 0

Exercise 88

Consider the system shown with PI control[13]



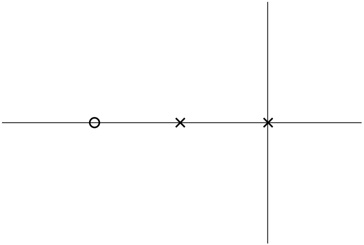
(a) Determine the transfer function from $Y(s)/R(s)$ and $Y(s)/W(s)$,

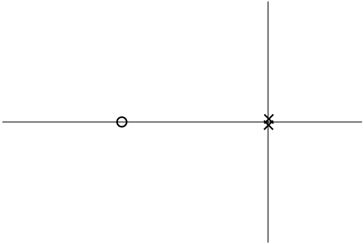
(b) Use Routh's criterion to find the range of k_p and k_i for which the system is stable.

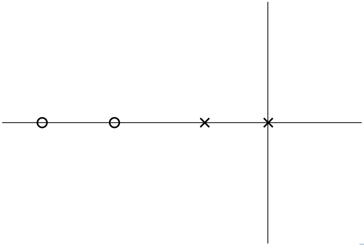
[13](b), $k_i > 0$ and $k_p > k_i - 2$

Exercise 89

Sketch the root locus







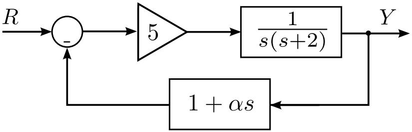
Exercise 89 - continued

To verify your results using Matlab, copy and past the following code

```
s = tf([1 0],[1]);
figure
rlocus((s+10)/(s*(s+5)))
figure
rlocus((s+5)/(s*s))
figure
rlocus((s+10)*(s+8)/(s*(s+4)))
```

Exercise 90

Sketch the root locus with respect to the parameter α, estimate the closed-loop pole locations, and sketch the corresponding step responses when $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 2$. Use Matlab to check the accuracy of your approximate step responses [14].



[14] The characteristic equation is $1 + \alpha \frac{5s}{s^2+2s+5}$

Exercise 91

A control system for positioning the head of a floppy disk drive has the closed-loop transfer function

$$T(s) = 11.1 \frac{s+18}{(s+20)(s^2+4s+10)}.$$

Plot the poles and zeros of this system and discuss the dominance of the complex poles. What percentage overshoot for a step input do you expect? Compare the results with the actual response using Matlab.[15]

[15] Dominant poles: 7.69%, actual overshoot 8%

Exercise 92

A unit feedback control system has the loop transfer function

$$L(s) = k\frac{s^2 + 10s + 30}{s^2(s + 10)}.$$

We desire the dominant roots to have a damping ratio of $\zeta = 0.707$. Find the gain k when this condition is satisfied. Use Matlab. [16]

[16] $k = 16$

Next class...

- Frequency response: Bode plots